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Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications

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Abstract

This paper is part II of a series of papers on the deformation quantization on the cotangent bundle of an arbitrary manifold Q. For certain homogeneous star products of Weyl ordered type (which we have obtained from a Fedosov type procedure in part I, see [M. Bordemann, N. Neumaier, S. Waldmann, Homogeneous Fedosov star products on cotangent bundles I: Weyl and standard ordering with differential operator representation, Comm. Math. Phys. 198 (1998) 363–396]) we construct differential operator representations via the formal GNS construction (see [M. Bordemann, S. Waldmann, Formal GNS construction and states in deformation quantization, Comm. Math. Phys. 195 (1998) 549–583]). The positive linear functional is integration over Q with respect to some fixed density and is shown to yield a reasonable version of the Schrödinger representation where a Weyl ordering prescription is incorporated. Furthermore we discuss simple examples like free particle Hamiltonians (defined by a Riemannian metric on Q) and the implementation of certain diffeomorphisms of Q to unitary transformations in the GNS (pre-)Hilbert space and of time reversal maps (involutive anti-symplectic diffeomorphisms of T^*Q) to anti-unitary transformations. We show that the fixed-point set of any involutive time reversal map is either empty or a Lagrangian submanifold. Moreover, we compare our approach to concepts using integral formulas of generalized Moyal–Weyl type. Furthermore we show that the usual WKB expansion with respect to a projectable Lagrangian submanifold can be formulated by a GNS construction. Finally we prove that any homogeneous star product on any cotangent bundle is strongly closed, i.e. the integral over T^*Q w.r.t. the symplectic volume vanishes on star-commutators. An alternative Fedosov type deduction of the star product of standard ordered type using a deformation of the algebra of symmetric contravariant tensor fields is given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Deformation quantization is a now well-established quantization concept introduced by Bayen et al.[4]. For any symplectic manifold the existence of the formal associative deformation (the star product) of the pointwise multiplication of smooth functions which are identified with the classical observables had been shown by DeWilde et al. [14] and Fedosov [18]. A recent preprint by Kontsevich [22] states that such star products even exist for any Poisson manifold. The classification for star products on symplectic manifolds up to equivalence transformations by formal power series in the second de Rham cohomology group is due to Refs. [5,23,24].

The particular case of a cotangent bundle T^*Q where Q is the physical configuration space is of great importance for physicists and there is a large amount of literature concerning various ways of quantization of such physical phase spaces. Deformation quantization and star products on cotangent bundles are considered e. g. in [12,13,25]. Moreover, differential operator representations and their symbolic calculus including integration techniques are considered in e.g. [16,17,29,30]. For geometric quantization of cotangent bundles see e.g. [31] and references therein.

In order to formulate states and Hilbert space representations of the deformed algebra of formal power series of smooth complex-valued functions on a symplectic manifold two of us (MB and SW) have recently transferred the concept of GNS representations in the theory of C^*-algebras to deformation quantization [9]. In that paper this concept was shown to be physically reasonable by applying it mainly to flat $\mathbb{R}^{2n}$: integration over configuration space at fixed momentum value 0 turned out to be a formally positive linear functional whose GNS representation yielded the usual formal Schrödinger representation as formal differential operators on $\mathbb{R}^n$ by means of the Weyl ordering prescription (see [8] for details). In another paper these results had been extended to the following situation: integration over a projectable Lagrangian submanifold L of $\mathbb{R}^{2n}$ with respect to a geometrically defined volume preceded by a geometrically defined formal series of differential operators yielded a formally positive linear functional whose GNS representation contained the usual WKB expansion of an eigenfunction of some Hamiltonian operator to its eigenvalue E provided L is contained in the energy surface of the classical Hamiltonian to the value E [10].

This paper is part II of a series of papers devoted to the study of certain star products on the cotangent bundle of an arbitrary manifold Q and applications of the GNS construction for this particular physically important situation. In part I (see [8]) we had prepared the ground by constructing a star product of standard ordered type $*_S$ (in which the first function is differentiated in vertical 'momentum' directions only) and a representation of this algebra where those functions on T^*Q canonically corresponding to symmetric contravariant tensor fields T on Q (i.e. which are polynomial in the momenta) are mapped to differential operators on Q where T is naturally paired with a multiple covariant derivative with respect to some torsion-free connection on Q. This had been done by a Fedosov type construction. Moreover, we had constructed a star product $*_{\mathrm{W}}$ of Weyl type on T^*Q equivalent to $*_{\mathrm{S}}$ and a *-representation (with respect to complex conjugation) of this algebra as generalized Weyl ordered differential operators.

The aim of this paper is to generalize the results of [9,10] to arbitrary cotangent bundles, that is to obtain the Schrödinger-like differential operator representation as well as the WKB expansion as GNS representations. Moreover, we discuss some simple applications and compare our results with the approaches using integral formulas like for instance Underhill's quantization [29] and [16,17,26] and give an elementary proof that any homogeneous star product on any cotangent bundle is strongly closed in the sense of [12].

The paper is organized as follows: Section 2 contains notation and results of the first part. In Section 3 we give another independent construction for the star product $*_{\mathrm{S}}$ which provides another way to show that this star product is differential. In Section 4 we consider a linear functional which is integration over the configuration space Q with respect to some chosen positive density and show that this turns out to be a formally positive linear functional. Hence it induces a GNS representation which is explicitly computed. In Section 5 we mention some easy but physically important convergence properties and discuss simple physical examples. Moreover, we consider particular symmetries of T^*Q and their implementation as automorphisms of the star product algebras and as unitary maps in the GNS Hilbert spaces. We prove that involutive anti-symplectic diffeomorphisms have either no fixed points or the fixed-point set is a Lagrangian submanifold and describe such maps for certain coadjoint orbits. In Section 6 we compare our formulas using integral formulas for the representations with other approaches already mentioned. Section 7 is devoted to the WKB expansion where we generalize the previous results to the case of projectable Lagrangian submanifolds in a cotangent bundle. Finally we prove in Section 8 that any homogeneous star product on T^*Q is strongly closed.

2. Preliminary results and notation

In this section we shall remember some results from [8,9] and establish our notation: Let Q be a smooth n-dimensional manifold and $\pi : T^*Q \to Q$ its cotangent bundle with the canonical one-form θ_0 and the canonical symplectic form $\omega_0 = -\,\mathrm{d}\theta_0$. Moreover, we denote by ξ the canonical Liouville vector field defined by $i_\xi \omega_0 = -\theta_0$. Let $i : Q \to T^*Q$ be the canonical embedding of Q as zero section. Moreover, we shall fix a torsion-free connection ∇ on Q and make use of local bundle Darboux coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ induced by local coordinates $q^1, \ldots, q^n$ on Q. A function $f \in C^\infty(T^*Q)$ is called polynomial in the momenta of degree k if $\mathcal{L}_\xi f = kf$ and the set of polynomial functions in the momenta of degree k is denoted by $C^\infty_{pp,k}(T^*Q)$ and we set $C^\infty_{pp}(T^*Q) := \bigoplus_{k=0}^\infty C^\infty_{pp,k}(T^*Q)$. Then $C^\infty_{pp}(T^*Q)$ is canonically isomorphic as graded algebra to $\Gamma(\bigvee TQ)$ together with the symmetric product $\vee$ via the canonical isomorphism $\Gamma(\bigvee^k TQ) \ni T \mapsto \widehat{T} \in C^\infty_{pp,k}(T^*Q)$

given by $\widehat{T}(\alpha_q) := (1/k!)T(\alpha_q, \ldots, \alpha_q)$ where $\alpha_q \in T_q^*Q$. For some easy homogeneity properties of $C^\infty_{pp}(T^*Q)$ see e.g. [8, Lemma A.2]. We shall use Einstein's summation convention, i.e. summation over repeated indices is understood.

Throughout this paper we denote the formal deformation parameter by λ which corresponds directly to Planck's constant $\hbar$. Using the connection ∇ on Q we had constructed a star product of standard ordered type $*_{\mathrm{S}}$ in [8] which is the natural generalization of the standard ordered product in flat $\mathbb{R}^{2n}$. A more physically star product of Weyl type having the complex conjugation as involutive antilinear anti-automorhism has been shown to exist and is constructed by the following equivalence transformation: we consider the operator

$$\Delta := \partial_{p_i}\partial_{q^i} + p_r(\pi^*\Gamma^r_{ij})\partial_{p_i}\partial_{p_j} + (\pi^*\Gamma^i_{ij})\partial_{p_j} + (\pi^*\alpha_j)\partial_{p_j}, \tag{1}$$

where locally $\alpha = \alpha_j\,\mathrm{d}q^j$ is a one-form satisfying $\mathrm{tr}R = -\mathrm{d}\alpha$ where R is the curvature tensor of ∇. This operator is globally defined and induces the equivalence transformation (see [8, Section 7])

$$N := \mathrm{e}^{(\lambda/2\mathrm{i})\Delta}, \tag{2}$$

which yields the star product $*_{\mathrm{W}}$ by

$$f *_{\mathrm{W}} g := N^{-1}((Nf) *_{\mathrm{S}} (Ng)), \tag{3}$$

where $f, g \in C^\infty(T^*Q)[[\lambda]]$. This equivalence is again the natural generalization of the flat case and hence we shall call $*_{\mathrm{W}}$ the star product of Weyl ordered type. Both star products have the important property of being homogeneous in the sense of [13]: the following operator

$$\mathcal{H} := \lambda\frac{\partial}{\partial\lambda} + \mathcal{L}_\xi \tag{4}$$

is a derivation of $*_{\mathrm{S}}$ and $*_{\mathrm{W}}$. Furthermore a representation on $C^\infty(Q)[[\lambda]]$ for the standard ordered product $*_{\mathrm{S}}$ was shown to be given by the following explicit formula:

$$\begin{aligned}\varrho_{\mathrm{S}}(f)\psi &:= i^*(f *_{\mathrm{S}} \pi^*\psi)\\ &= \sum_{r=0}^\infty \frac{1}{r!}\left(\frac{\lambda}{\mathrm{i}}\right)^r i^*\left(\frac{\partial^r f}{\partial p_{i_1}\cdots\partial p_{i_r}}\right) i_s(\partial_{q^{i_1}})\cdots i_s(\partial_{q^{i_r}})\frac{1}{r!}D^r\psi,\end{aligned} \tag{5}$$

where $f \in C^\infty(T^*Q)[[\lambda]]$ and $\psi \in C^\infty(Q)[[\lambda]]$ and where $D := dq^k \vee \nabla_{\partial_{q^k}}$ justifying the notion of 'standard ordered type' by analogy to the flat case. Then a representation ϱ_{W} for $*_{\mathrm{W}}$ on the same representation space is given by

$$\varrho_{\mathrm{W}}(f)\psi := \varrho_{\mathrm{S}}(Nf)\psi = i^*((Nf) *_{\mathrm{S}} \pi^*\psi), \tag{6}$$

yielding the explicit formula

$$\varrho_{\mathrm{W}}(f)\psi = \sum_{r=0}^\infty \frac{1}{r!}\left(\frac{\lambda}{\mathrm{i}}\right)^r i^*\left(\frac{\partial^r Nf}{\partial p_{i_1}\cdots\partial p_{i_r}}\right) i_s(\partial_{q^{i_1}})\cdots i_s(\partial_{q^{i_r}})\frac{1}{r!}D^r\psi. \tag{7}$$

In [9] a generalization of the well-known GNS construction for C^*-algebras to deformation quantization was proposed where the crucial ingredient is the notion of positivity in the ring of formal power series $\mathbb{R}[[\lambda]]$ (see e.g. Appendix A). We have argued that it is advantageous to consider the field of formal Laurent series instead of $\mathbb{R}[[\lambda]]$ which is in a canonical way an ordered field. Moreover, it turned out that certain field extensions (the formal NP and CNP series) are even more suitable for the definition of the GNS representation and 'formal' Hilbert spaces. In Appendix A we provide several lemmas and definitions how the results obtained for the more convenient formal power series can be extended to these more general formal series justifying thereby our sloppy usage of the notion of 'Hilbert' spaces, etc. In fact all relevant structures are completely determined in the setting of formal power series, see Lemma A.4 and A.5.

3. Another Fedosov-like construction of the standard ordered star product $*_S$

In this section we shall give a different method to obtain the standard ordered star product $*_S$ as in [8] avoiding the additional technical ingredient of a lifted connection. To do so we first notice that it is enough to consider $C^\infty_{pp}(T^*Q)[[\lambda]]$ instead of $C^\infty(T^*Q)[[\lambda]]$ since the star product is given by bidifferential operators, which are uniquely determined by their values on $C^\infty_{pp}(T^*Q)$. Using the canonical algebra isomorphism $\widehat{\ }$ between $\Gamma(\bigvee TQ)$ and $C^\infty_{pp}(T^*Q)$ the task of finding a deformation of the pointwise multiplication in $C^\infty_{pp}(T^*Q)$ is equivalent to the deformation of the $\vee$-product in the symmetric algebra $\Gamma(\bigvee TQ)$. Therefore we start a Fedosov procedure similar to the construction in [6] to deform the $\vee$-product using a connection on Q and show that this deformation is compatible to the standard ordered representation of functions that are polynomial in the momentum variables, giving an alternative method to construct the star product $*_S$.

3.1. *A deformation $\star_S$ of the $\vee$-product*

We consider the slightly modified Fedosov algebra

$$\mathcal{W} \otimes \bigvee \otimes \Lambda := \left(\mathsf{X}_{s=0}^{\infty} \mathbb{C}\left(\Gamma\left(\bigvee^s T^*Q \otimes \bigvee TQ \otimes \bigwedge T^*Q\right)\right)\right)[[\lambda]].$$

To avoid clumsy notation we drop the explicit mention of the complexification that shall be taken for granted. To define the important mappings we make use of factorized sections of the shape $F_i = \lambda^{q_i} f_i \otimes S_i \otimes \alpha_i$ with $f_i \in \Gamma(\bigvee^{s_i} T^*Q)$, $S_i \in \Gamma(\bigvee^{d_i} TQ)$, $\alpha_i \in \Gamma(\bigwedge^{a_i} T^*Q)$. In addition we consider the obvious degree maps $\deg_s$, $\deg_s^*$, $\deg_a$, and $\deg_\lambda$ with

$$\deg_s F_i = s_i F_i, \quad \deg_s^* F_i = d_i F_i, \quad \deg_a F_i = a_i F_i, \quad \deg_\lambda F_i = q_i F_i.$$

The additional symmetric degree with respect to $\deg_s^*$ is referred to as *dual* symmetric degree. The undeformed multiplication in $\mathcal{W} \otimes \bigvee \otimes \Lambda$ of two sections F_1, F_2 is defined by

$$\mu \circ (F_1 \otimes F_2) = F_1 F_2 := \lambda^{q_1+q_2} (f_1 \vee f_2) \otimes (S_1 \vee S_2) \otimes (\alpha_1 \wedge \alpha_2).$$

By σ we denote the linear map $\sigma : \mathcal{W} \otimes \bigvee \otimes \Lambda \to \Gamma(\bigvee TQ)[[\lambda]]$ that projects onto the part of symmetric and anti-symmetric degree zero. Given a torsion-free connection ∇ on Q we define the connection ∇ in $\mathcal{W} \otimes \bigvee \otimes \Lambda$ using a chart $q^1, \ldots, q^n$ of Q by

$$\nabla := (1 \otimes 1 \otimes \mathrm{d}q^l)\nabla_{\partial_{q^l}},$$

which turns out to be a globally defined homogeneous super derivation of the pointwise product of degree $(0, 0, 1)$. Now we define the fibrewise associative deformed multiplication $\circ_S$ for $F, G \in \mathcal{W} \otimes \bigvee \otimes \Lambda$ by

$$F \circ_S G := \mu \circ \mathrm{e}^{(\lambda/\mathrm{i}) i_s^*(\mathrm{d}q^l) \otimes i_s(\partial_{q^l})} F \otimes G,$$

where $i_s^*(\mathrm{d}q^l)$ denotes the symmetric insertion of the one-form $\mathrm{d}q^l$ and $i_s(\partial_l)$ denotes the symmetric insertion of the vector field ∂_{q^l}. Following Fedosov we define the mappings

$$\delta := (1 \otimes 1 \otimes \mathrm{d}q^l) i_s(\partial_{q^l}) \quad \text{and} \quad \delta^* := (\mathrm{d}q^l \otimes 1 \otimes 1) i_a(\partial_{q^l}),$$

which are super derivations of the undeformed product of degree $(-1, 0, 1)$ and $(1, 0, -1)$ and for which the following identities hold:

$$\delta^2 = \delta^{*2} = 0, \quad \delta\delta^* + \delta^*\delta = \deg_s + \deg_a, \quad \nabla\delta + \delta\nabla = 0.$$

In addition one easily verifies that δ resp. ∇ are super derivations with respect to $\circ_S$ of degree $(-1, 0, 1)$ resp. $(0, 0, 1)$. In analogy to the known Fedosov construction we define

$$\delta^{-1} F_1 := \begin{cases} \dfrac{1}{s_1 + a_1} \delta^* F_1 & \text{for } s_1 + a_1 \neq 0, \\ 0 & \text{for } s_1 + a_1 = 0, \end{cases}$$

with which a 'Hodge-decomposition' for any $F \in \mathcal{W} \otimes \bigvee \otimes \Lambda$ is valid, i.e.

$$F = \delta\delta^{-1} F + \delta^{-1}\delta F + \sigma(F).$$

Finally we define $\deg_a$-graded super commutators with respect to $\circ_S$ by $[F_1, F_2] := \mathrm{ad_S}(F_1) F_2 := F_1 \circ_S F_2 - (-1)^{a_1 a_2} F_2 \circ_S F_1$ and the total degree map $\mathrm{Deg} := 2\deg_\lambda + \deg_s + \deg_s^*$ being a $\circ_S$ super derivation of degree $(0, 0, 0)$. The operator corresponding to the homogeneity derivation $\mathcal{H}$ as in (4) is given by $\mathsf{H} := \deg_s^* + \deg_\lambda$ and is a (only $\mathbb{C}$-linear!) derivation of $\circ_S$. After these preparations we have the first little lemma being of importance for the construction of the Fedosov derivation:

Lemma 3.1. *For $R_S := -\frac{1}{2} R^l_{kij}\, \mathrm{d}q^k \otimes \partial_{q^l} \otimes \mathrm{d}q^i \wedge \mathrm{d}q^j$ denoting by R^l_{kij} the components of the curvature tensor of the connection ∇ we have*

$$\nabla^2 = \frac{\mathrm{i}}{\lambda} \mathrm{ad_S}(R_S), \qquad \delta R_S = 0, \qquad \nabla R_S = 0.$$

Proof. The first assertion is a straightforward computation and the others are just reformulations of the Bianchi-identities. □

For the Fedosov derivation $\mathcal{D}_{\mathrm{S}}$ satisfying $\mathcal{D}_{\mathrm{S}}^2 = 0$ we make the usual Ansatz

$$\mathcal{D}_{\mathrm{S}} = -\delta + \nabla + \frac{\mathbf{i}}{\lambda}\mathrm{ad}_{\mathrm{S}}(r_{\mathrm{S}}),$$

with $r_{\mathrm{S}} \in \mathcal{W} \otimes \bigvee \otimes \Lambda^1$ (i.e. $\deg_a r_{\mathrm{S}} = r_{\mathrm{S}}$) and completely analogously to the usual Fedosov procedure (see e.g. [8,18]) we obtain a uniquely determined such r_{S} obeying the conditions $\delta^{-1} r_{\mathrm{S}} = 0$ and $\delta r_{\mathrm{S}} = \nabla r_{\mathrm{S}} + R_{\mathrm{S}} + (\mathbf{i}/\lambda) r_{\mathrm{S}} \circ_{\mathrm{S}} r_{\mathrm{S}}$. Moreover, $r_{\mathrm{S}} = \sum_{k=3}^{\infty} r_{\mathrm{S}}^{(k)}$ with $\mathrm{Deg}\, r_{\mathrm{S}}^{(k)} = k r_{\mathrm{S}}$ is given by the following recursion formula:

$$r_{\mathrm{S}}^{(3)} = \delta^{-1} R_{\mathrm{S}}, \qquad r_{\mathrm{S}}^{(k+3)} = \delta^{-1}\left(\nabla r_{\mathrm{S}}^{(k+2)} + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{k-1} r_{\mathrm{S}}^{(l+2)} \circ_{\mathrm{S}} r_{\mathrm{S}}^{(k-l+2)}\right).$$

Using this recursion formula one can show by induction on the total degree that $\mathsf{H} r_{\mathrm{S}} = r_{\mathrm{S}}$ and hence $\mathsf{H}\mathcal{D}_{\mathrm{S}} = \mathcal{D}_{\mathrm{S}}\mathsf{H}$ implying together with the particular shape of $\circ_{\mathrm{S}}$ the following easier recursion formula

$$r_{\mathrm{S}}^{(3)} = \delta^{-1} R_{\mathrm{S}}, \qquad r_{\mathrm{S}}^{(k+3)} = \delta^{-1}\left(\nabla r_{\mathrm{S}}^{(k+2)} - \frac{1}{2}\sum_{l=1}^{k-1} \{r_{\mathrm{S}}^{(l+2)}, r_{\mathrm{S}}^{(k-l+2)}\}_{\mathrm{fib}}\right),$$

denoting by $\{\cdot,\cdot\}_{\mathrm{fib}}$ the fibrewise Poisson bracket given by $\{F, G\}_{\mathrm{fib}} = i_s(\partial_{q^l}) F i_s^*(\mathrm{d}q^l) G - i_s^*(\mathrm{d}q^l) F i_s(\partial_{q^l}) G$. Obviously this implies that r_{S} does not depend on λ at all. Moreover, we have the following proposition:

Proposition 3.2. *$\mathcal{W}_{\mathcal{D}_{\mathrm{S}}} := ker(\mathcal{D}_{\mathrm{S}}) \cap \mathcal{W} \otimes \bigvee$ is a subalgebra of $(\mathcal{W} \otimes \bigvee \otimes \Lambda, \circ_{\mathrm{S}})$ and the map σ restricted to $\mathcal{W}_{\mathcal{D}_{\mathrm{S}}}$ is a $\mathbb{C}[[\lambda]]$-linear bijection onto $\Gamma(\bigvee TQ)[[\lambda]]$. Denoting by $\tau_{\mathrm{S}} : \Gamma(\bigvee TQ)[[\lambda]] \to \mathcal{W}_{\mathcal{D}_{\mathrm{S}}} \subset \mathcal{W} \otimes \bigvee$ the inverse of the restriction of σ to $\mathcal{W}_{\mathcal{D}_{\mathrm{S}}}$ we have the following recursion scheme to calculate $\tau_{\mathrm{S}}(T)$ for $T = \sum_{t=0}^{m} T^{(t)}$ ($m \in \mathbb{N}$) and $T^{(t)} \in \Gamma(\bigvee^t TQ)$: The Fedosov–Taylor series $\tau_{\mathrm{S}}(T) = \sum_{l=0}^{\infty} \tau_{\mathrm{S}}(T)^{(l)}$ with* $\mathrm{Deg}\, \tau_{\mathrm{S}}(T)^{(l)} = l \tau_{\mathrm{S}}(T)^{(l)}$ *is given by*

$$\tau_{\mathrm{S}}(T)^{(0)} = T^{(0)}, \tag{8}$$

$$\tau_{\mathrm{S}}(T)^{(k+1)} = \delta^{-1}\left(\nabla \tau_{\mathrm{S}}(T)^{(k)} + \frac{\mathbf{i}}{\lambda}\sum_{t=1}^{k-1} \mathrm{ad}_{\mathrm{S}}(r_{\mathrm{S}}^{(t+2)}) \tau_{\mathrm{S}}(T)^{(k-t)}\right) + T^{(k+1)} \quad \text{for } 0 \le k \le m-1, \tag{9}$$

$$\tau_{\mathrm{S}}(T)^{(k+1)} = \delta^{-1}\left(\nabla \tau_{\mathrm{S}}(T)^{(k)} + \frac{\mathbf{i}}{\lambda}\sum_{t=1}^{k-1} \mathrm{ad}_{\mathrm{S}}(r_{\mathrm{S}}^{(t+2)}) \tau_{\mathrm{S}}(T)^{(k-t)}\right) \quad \text{for } k \ge m. \tag{10}$$

In addition the $\mathbb{C}[[\lambda]]$-linear mapping τ_{S} satisfies

$$\mathsf{H}\tau_{\mathrm{S}}(T) = \tau_{\mathrm{S}}(\mathsf{H}T) \quad \forall T \in \Gamma(\bigvee TQ)[[\lambda]]. \tag{11}$$

Proof. The recursion formula is proven analogously to [8, Theorem 2.2] and the homogeneity of τ_{S} follows directly from $\mathsf{H}\mathcal{D}_{\mathrm{S}} = \mathcal{D}_{\mathrm{S}}\mathsf{H}$. □

Remark. Observing the homogeneity of τ_{S} with respect to H the recursion formula for $\tau_{\mathrm{S}}(T) = \sum_{k=0}^{\infty} \tau_{\mathrm{S}}(T)_{(k)}$ with $(\deg_s + \deg_\lambda)\tau_{\mathrm{S}}(T)_{(k)} = k\tau_{\mathrm{S}}(T)_{(k)}$ for $T \in \Gamma(\bigvee TQ)$ can be rewritten in a more convenient manner, i.e.

$$\tau_{\mathrm{S}}(T)_{(0)} = T, \qquad \tau_{\mathrm{S}}(T)_{(k+1)} = \delta^{-1}\left(\nabla\tau_{\mathrm{S}}(T)_{(k)} + \frac{\mathbf{i}}{\lambda}\sum_{t=1}^{k} \mathrm{ad}_{\mathrm{S}}(r_{\mathrm{S}}^{(t+2)})\tau_{\mathrm{S}}(T)_{(k-t)}\right).$$

Now the associative product $\star_{\mathrm{S}}$ for $\Gamma(\bigvee TQ)[[\lambda]]$ is defined by pull-back of $\circ_{\mathrm{S}}$ via τ_{S}, i.e.

$$S \star_{\mathrm{S}} T := \sigma(\tau_{\mathrm{S}}(S) \circ_{\mathrm{S}} \tau_{\mathrm{S}}(T))$$

for $S, T \in \Gamma(\bigvee TQ)[[\lambda]]$, that induces an associative product $*_{\mathrm{S}}$ for $C^{\infty}_{pp}(T^*Q)[[\lambda]]$ by pull-back via the natural isomorphism $\widehat{\ }$, i.e.

$$\widehat{S} *_{\mathrm{S}} \widehat{T} := (\widehat{S \star_{\mathrm{S}} T}) \tag{12}$$

for $\widehat{S}, \widehat{T} \in C^{\infty}_{pp}(T^*Q)[[\lambda]]$. We shall now prove that this definition indeed coincides with the standard ordered product as in [8]. Using (11) and the fact that H is a $\star_{\mathrm{S}}$-derivation one can show that $\mathcal{H}$ is a derivation of $*_{\mathrm{S}}$ and thus $*_{\mathrm{S}}$ is homogeneous.

3.2. A representation of $(\Gamma(\bigvee TQ)[[\lambda]], \star_{\mathrm{S}})$

This section is dedicated to construct a representation of $(\Gamma(\bigvee TQ)[[\lambda]], \star_{\mathrm{S}})$, that is compatible with the standard representation we constructed in [8, Section 6]. As a starting point we find a fibrewise representation of $(\mathcal{W} \otimes \bigvee, \circ_{\mathrm{S}})$. First we define the representation space

$$\mathfrak{H} := \mathcal{W} \otimes \bigvee \cap \ker(\deg_s^*)$$

and the projection $P : \mathcal{W} \otimes \bigvee \to \mathcal{W} \otimes \bigvee \cap \ker(\deg_s^*)$ that projects onto the subspace of dual symmetric degree zero. For $T \in \mathcal{W} \otimes \bigvee$ and $\Psi \in \mathfrak{H}$ we define the fibrewise standard representation $\widetilde{\rho}_{\mathrm{S}} : \mathcal{W} \otimes \bigvee \to \mathsf{End}(\mathfrak{H})$ by

$$\widetilde{\rho}_{\mathrm{S}}(T)\Psi := P(T \circ_{\mathrm{S}} \Psi). \tag{13}$$

Lemma 3.3. *The map $\widetilde{\rho}_{\mathrm{S}}$ is a $\circ_{\mathrm{S}}$-representation of $\mathcal{W} \otimes \bigvee$ on $\mathfrak{H}$, i.e. for $S, T \in \mathcal{W} \otimes \bigvee$ we have*

$$\widetilde{\rho}_{\mathrm{S}}(S \circ_{\mathrm{S}} T) = \widetilde{\rho}_{\mathrm{S}}(S)\widetilde{\rho}_{\mathrm{S}}(T).$$

Proof. The $\mathbb{C}[[\lambda]]$-lineartity of $\widetilde{\rho}_{\mathrm{S}}$ is obvious and the representation property is proved by straightforward computation using the associativity of $\circ_{\mathrm{S}}$ and the validity of the equation $P(F \circ_{\mathrm{S}} G) = P(F \circ_{\mathrm{S}} PG)$ for $F, G \in \mathcal{W} \otimes \bigvee$, which follows from the particular shape of $\circ_{\mathrm{S}}$. □

What we have in mind is to define a $\star_{\mathrm{S}}$-representation on the representation space $C^{\infty}(Q)[[\lambda]]$ that can canonically be identified with $\mathcal{W} \otimes \bigvee \cap \ker(\deg_s^*) \cap \ker(\deg_s)$, thus for $\psi \in C^\infty(Q)[[\lambda]]$ and $T \in \Gamma(\bigvee TQ)[[\lambda]]$ the following expression is well defined:

$$\rho_S(T)\psi := p(T \star_S \psi), \tag{14}$$

denoting by p the projection from $\Gamma(\bigvee TQ)[[\lambda]]$ to the part of dual symmetric degree zero, i.e. $C^\infty(Q)[[\lambda]]$. The representation property of ρ_S can now be easily proven:

Proposition 3.4. *Let $\mathcal{D}_S$ be the Fedosov derivation constructed above and let τ_S be the corresponding Fedosov–Taylor series and $\star_S$ the deformation of the $\vee$-product. Then*

$$\mathcal{D}_S P = P\mathcal{D}_S \tag{15}$$

and

$$\rho_S(T)\psi = p(T\star_S\psi) = \sigma(\widetilde{\rho_S}(\tau_S(T))\tau_S(\psi)) \tag{16}$$

defines a $\star_S$-representation of $\Gamma(\bigvee TQ)[[\lambda]]$ where $\psi \in C^\infty(Q)[[\lambda]]$ and $T \in \Gamma(\bigvee TQ)[[\lambda]]$.

Proof. As an abuse of notation we also denoted by P the mapping $P \otimes 1 : \mathcal{W} \otimes \bigvee \otimes \Lambda^1 \to \mathcal{W} \otimes \bigvee \otimes \Lambda^1 \cap \ker(\deg_s^*)$. Then one immediately checks that δ and ∇ commute with P. Moreover, $\mathrm{ad}_S(r_S)$ commutes with P due to the particular shape of $\circ_S$ and $\deg_s^* r_S = r_S$. Using this and the obvious equation $p(\sigma(F)) = \sigma(P(F))$ for $F \in \mathcal{W}$ it is straightforward proving the representation property of ρ_S observing Eq. (15) and $P(F \circ_S G) = P(F \circ_S PG)$ for $F, G \in \mathcal{W} \otimes \bigvee$. □

To conclude this section we shall now show the compatibility of the representation ϱ_S to the standard representation constructed in [8], implying that $*_S$ defined on $C^\infty_{pp}(T^*Q)[[\lambda]]$ by pull-back of $\star_S$ as above coincides with the standard ordered star product we considered in [8]. Moreover, the star product $*_S$ for $f, g \in C^\infty(T^*Q)[[\lambda]]$ is completely determined by this construction since bidifferential operators are completely determined by their values on functions which are polynomial in the momenta.

Proposition 3.5. *For $\psi \in C^\infty(Q)[[\lambda]]$ the Fedosov–Taylor series is given by*

$$\tau_S(\psi) = \mathrm{e}^D\psi \text{ with } D := \mathrm{d}q^l \vee \nabla_{\partial_{q^l}}. \tag{17}$$

Therefore the representation ρ_S can be calculated explicitly for $T \in \Gamma(\bigvee TQ)[[\lambda]]$ and is given by

$$\rho_S(T)\psi = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\lambda}{\mathrm{i}}\right)^r p(i_s^*(\mathrm{d}q^{i_1})\cdots i_s^*(\mathrm{d}q^{i_r})T) i_s(\partial_{q^{i_1}})\cdots i_s(\partial_{q^{i_r}})\frac{1}{r!}D^r\psi \tag{18}$$

and thus

$$\rho_S(T)\psi = \varrho_S(\widehat{T})\psi. \tag{19}$$

Hence the obvious injectivity of ρ_S implies that the product defined by (12) *indeed coincides with* $*_S$.

Proof. The formula for $\tau_S(\psi)$ can be proven by induction on the total degree using the recursion formula given in (8). Eq. (18) then follows by direct calculation from the definition of the representation ρ_S in Eq. (14). Clearly (19) follows by direct comparison with (5). □

At last in this section we should mention that our considerations yield an alternative proof of the fact that the standard representation for functions polynomial in the momenta gives rise to a standard ordered star product $*_S$ of Vey type since we have the following proposition:

Proposition 3.6. *For $S, T \in \Gamma(\bigvee TQ)$ there are bidifferential operators N_r ($r \in \mathbb{N}$) on $\Gamma(\bigvee TQ)$ that are of order r in both arguments such that $S \star_S T$ can be written as*

$$S \star_S T = \sum_{r=0}^{\infty} \left(\frac{\lambda}{\mathrm{i}}\right)^r \mathsf{N}_r(S, T).$$

Proof. The proof is an easy consequence of the definition of $\star_S$ and the fact that the mapping $\Gamma(\bigvee TQ) \ni T \mapsto \tau_S(T)_{(r),l} \in \Gamma(\bigvee^l T^*Q \otimes \bigvee TQ)$ is a differential operator of order r for all $0 \le l \le r$ (where we have written $\tau_S(T)_{(r)} = \sum_{l=0}^{r} (\lambda/\mathrm{i})^{r-l} \tau_S(T)_{(r),l}$), what can be proven by straightforward induction on r using the recursion formula for $\tau_S(T)_{(r)}$. □

Remark. At this instance we should mention that we use the notion of differential operators on the commutative, associative algebra $(\Gamma(\bigvee TQ), \vee)$ as a purely algebraic concept. For the reader unfamiliar with this point of view, we just give a short definition: By l_T we denote the left multiplication by $T \in \Gamma(\bigvee TQ)$ that is defined to be a differential operator of order 0. Recursively the differential operators of order $k \in \mathbb{N}$ are defined by the set of all endomorphisms $\mathcal{D}$ of $\Gamma(\bigvee TQ)$ satisfying that $[\mathcal{D}, l_T]$ is a differential operator of order $k - 1$. Similarly one defines multi-differential operators between modules over a commutative and associative algebra. In view of this definition the insertions $i_s^*(\beta)$ for a one-form β have to be considered as differential operators of order 1 on $\Gamma(\bigvee TQ)$ since for all $T \in \Gamma(\bigvee TQ)$ we have $[i_s^*(\beta), l_T] = l_{i_s^*(\beta)T}$.

Moreover, the conjugation with the natural algebra-isomorphism $\widehat{\ }$ leaving invariant the order of differential operators yields bidifferential operators N_r of order r in every argument on $C^\infty_{pp}(T^*Q)$ given by $N_r(\widehat{S}, \widehat{T}) = \widehat{\mathsf{N}_r(S, T)}$ implying that the star product $*_S$ is of Vey type.

4. A GNS construction for a Schrödinger representation

For a given complex number γ let $|\bigwedge|^\gamma T^*Q$ be the bundle of γ-densities on Q (see e.g. [3, pp. 119–121]): this bundle can be obtained by taking the bundle of linear frames

$L(Q)$ over Q and by associating the typical fibre $\mathbb{C}$ by means of the Lie group action of the structure group $GL(n,\mathbb{R})$ on $\mathbb{C}$ given by $(g,z) \mapsto |\det(g)|^{\gamma} z$. The smooth sections of $|\bigwedge|^{\gamma} T^*Q$ are called complex γ-densities. In case γ is real the bundle $|\bigwedge|^{\gamma} T^*Q$ contains an obvious real subbundle (where the above action of the general linear group is restricted to the real numbers in $\mathbb{C}$) which we shall call the bundle of real γ-densities. The particular cases $\gamma = 1$ and $\gamma = \frac{1}{2}$, respectively, are called the bundle of (real or complex) densities and half-densities, respectively. By a standard partition of unity argument there always exist nonvanishing sections of the bundle of real densities and hence the density bundles are trivial. Let us fix once and for all an arbitrary nonvanishing real positive density μ on Q. This density locally defines a Riemann integral $\phi \mapsto \int \phi \mu_{1\ldots n}\, \mathrm{d}q^1 \cdots \mathrm{d}q^n$ for any continuous real-valued function ϕ with support in a coordinate neighbourhood U in which μ takes the local form $\mu_{1\ldots n}\, \mathrm{d}q^1 \cdots \mathrm{d}q^n$, which is extended to all of Q by means of the usual partition of unity argument and Riesz' theorem to a Lebesgue integral on Q.

The connection ∇ now defines a unique one-form α by

$$\nabla_X \mu =: \alpha(X)\mu \tag{20}$$

for an arbitrary vector field X on Q where the extension of ∇ to the bundle of γ-densities is obvious. For any two vector fields X, Y on Q we get upon evaluating $\nabla_X \nabla_Y \mu - \nabla_Y \nabla_X \mu - \nabla[X,Y]\mu$ the formula

$$\mathrm{tr} R(X,Y) = -\,\mathrm{d}\alpha(X,Y). \tag{21}$$

Thus having fixed the one-form α we can now take the operator N as in (2) and pass to the star product of Weyl type $*_{\mathrm{W}}$ given by (3).

Let $C^{\infty}_{Q}(T^*Q)[[\lambda]]$ denote the space of all formal series in λ whose coefficients lie in the space of all those smooth complex-valued functions on T^*Q whose support intersected with (the zero-section) Q is compact. This is clearly a two-sided ideal of $(C^{\infty}(T^*Q)[[\lambda]], *_{\mathrm{W}})$ stable under complex conjugation. Moreover, let $C^{\infty}_{0}(Q)[[\lambda]]$ denote the space of all formal series in λ whose coefficients are smooth complex-valued functions of compact support. This space carries a $\mathbb{C}[[\lambda]]$-sesquilinear form defined by

$$\langle \phi, \psi \rangle := \int_Q \overline{\phi}\psi\, \mu. \tag{22}$$

Lemma 4.1. *The formal series of differential operators corresponding to the standard and Weyl representation ϱ_{S} and ϱ_{W} enjoy the following symmetry properties for $f \in C^{\infty}(T^*Q)[[\lambda]]$ and $\phi, \psi \in C^{\infty}_{0}(Q)[[\lambda]]$:*

$$\langle \varrho_{\mathrm{S}}(f)\phi, \psi \rangle = \langle \phi, \varrho_{\mathrm{S}}(N^2\overline{f})\psi \rangle \quad \textit{and} \quad \langle \varrho_{\mathrm{W}}(f)\phi, \psi \rangle = \langle \phi, \varrho_{\mathrm{W}}(\overline{f})\psi \rangle. \tag{23}$$

Proof. The first equation is proved by integration by parts analogously to the proof given in [8, Theorem 7.3] where this equation had been proved for functions ϕ, ψ having their support in a coordinate neighbourhood. By writing the coefficients of ϕ and ψ in each order of λ as finite linear combinations of smooth complex-valued functions with support in coordinate neighbourhoods of an atlas and by (anti-) linearity the local proof can be extended to the above statement. Then the second equation is an immediate consequence of the first and the definition of ϱ_{W}. □

We are now ready to define and discuss a GNS construction analogous to the one considered for flat $\mathbb{R}^{2n}$ in [9, Proposition 11]. The positive functional which we want to consider is just the integration over the configuration space with respect to the fixed densitiy μ:

Proposition 4.2. *Let $\omega_\mu : C^\infty_Q(T^*Q)[[\lambda]] \to \mathbb{C}[[\lambda]]$ be the following $\mathbb{C}[[\lambda]]$-linear functional*

$$\omega_\mu(f) := \int_Q i^* f \, \mu, \tag{24}$$

then ω_μ is well defined and we have

$$\omega_\mu(Nf) = \omega(f), \tag{25}$$

$$\omega_\mu(f *_{\mathrm{W}} g) = \int_Q i^*(N^{-1} f) i^*(Ng) \, \mu. \tag{26}$$

It follows that ω_μ is formally positive, i.e.

$$\omega_\mu(\overline{f} *_{\mathrm{W}} f) = \int_Q i^*(\overline{Nf}) i^*(Nf) \, \mu \geq 0 \tag{27}$$

and its Gel'fand ideal is given by

$$\mathcal{J}_\mu = \{ f \in C^\infty_Q(T^*Q)[[\lambda]] \mid i^* N f = 0 \}. \tag{28}$$

Proof. To prove (25)–(27) it is sufficient to consider functions in $f, g \in C^\infty_Q(T^*Q)$ due to Lemma A.5. Since $i^* f$ has compact support it follows that ω_μ is well defined. Let χ be a smooth function with values in [0, 1] and compact support on Q which is equal to 1 on an open neighbourhood containing the support of $i^* f$. Clearly,

$$i^* f = \chi i^* f \chi = \chi i^*(f \pi^* \chi) = \chi i^*(f *_{\mathrm{S}} \pi^* \chi) = \chi \varrho_{\mathrm{S}}(f)(\chi),$$

where the second to last equality follows from the fact that the vertical derivatives of f at the zero section vanish outside the intersection of the support of f with Q whereas the covariant derivatives of χ vanish on the intersection of the support of f with Q. Integrating the above equation over Q with respect to μ and using Lemma 4.1 we get

$$\omega_\mu(f) = \langle \chi, \varrho_{\mathrm{S}}(f)\chi \rangle = \langle \varrho_{\mathrm{S}}(N^2 \overline{f})\chi, \chi \rangle = \int_Q \varrho_{\mathrm{S}}(N^{-2} f)(\chi) \chi \, \mu$$

$$= \int_Q i^*(N^{-2} f) \, \mu = \omega_\mu(N^{-2} f).$$

Hence (using the fact that N is invertible) $\omega_Q((N-\mathsf{id})(N+\mathsf{id})f)=0$ and since N starts with id it follows that $\mathsf{id}+N$ is still invertible on $C_Q^\infty(T^*Q)[[\lambda]]$ whence $\omega_\mu((N-\mathsf{id})f)=0$ for all $f\in C_Q^\infty(T^*Q)[[\lambda]]$ which proves (25). For Eq. (26) we get using the first one and the definition of $*_{\mathrm{W}}$ that

$$\begin{aligned}\omega_\mu(f *_{\mathrm{W}} g) &= \int_Q i^*((Nf) *_{\mathrm{S}} (Ng))\chi\chi\,\mu = \int_Q i^*((Nf) *_{\mathrm{S}} (Ng)\pi^*\chi)\chi\,\mu\\ &= \int_Q i^*(((Nf) *_{\mathrm{S}} (Ng)) *_{\mathrm{S}} \pi^*\chi)\chi\,\mu = \langle \chi, \varrho_{\mathrm{S}}((Nf) *_{\mathrm{S}} (Ng))\chi\rangle\\ &= \langle \varrho_{\mathrm{S}}(Nf)^\dagger\chi, \varrho(Ng)\chi\rangle = \langle \varrho_{\mathrm{S}}(N\overline{f})\chi, \varrho(Ng)\chi\rangle\\ &= \int_Q i^*(N^{-1}f)\chi i^*(Ng)\chi\,\mu = \int_Q i^*(N^{-1}f)i^*(Ng)\,\mu,\end{aligned}$$

where χ was chosen as above obeying the condition to be equal to one on an open set in Q containing the union of the supports of i^*f and i^*g. Then Eq. (27) is an obvious particular case of Eq. (26) and implies that $\omega_\mu(\overline{f} *_{\mathrm{W}} f)=0$ iff $i^*Nf=0$ vanishes since the integrand on the right-hand side of Eq. (27) is formally positive: this proves the simple formula for the Gel'fand ideal of ω_μ. □

Now we use the positive functional ω_μ to define a GNS representation in the following standard way: Let $\mathfrak{H}_\mu := C_Q^\infty(T^*Q)[[\lambda]]/\mathcal{J}_\mu$ be the GNS (pre-) Hilbert space and denote the equivalence class in $\mathfrak{H}_\omega$ of a function f by ψ_f. Then the $\mathbb{C}[[\lambda]]$-valued Hermitian product in $\mathfrak{H}_\mu$ is given by $\langle\psi_f,\psi_g\rangle := \omega_\mu(\overline{f} *_{\mathrm{W}} g)$ and the representation $\pi_\mu(f)$ is determined by $\pi_\mu(f)\psi_g := \psi_{f *_{\mathrm{W}} g}$. First we notice that the representation π_μ is not only defined for functions in $C_Q^\infty(T^*Q)[[\lambda]]$ but for all functions $C^\infty(T^*Q)[[\lambda]]$ since $\mathcal{J}_\mu$ is even a left ideal in $C^\infty(T^*Q)[[\lambda]]$ which can be verified directly using the Cauchy–Schwartz inequality and the fact that $C_Q^\infty(T^*Q)[[\lambda]]$ is a two-sided ideal stable under complex conjugation. Now we can easily state the main theorem of this section characterizing the representation more explicitly:

Theorem 4.3. *With the notations from above we have:*

(i) *The representation space $\mathfrak{H}_\mu$ is canonically isometric to $C_0^\infty(Q)[[\lambda]]$ equipped with the $\mathbb{C}[[\lambda]]$-valued Hermitian product* (22) *via the isometric isomorphism*

$$\Phi:\psi_f\mapsto i^*Nf \quad \text{and its inverse} \quad \Phi^{-1}:\chi\mapsto\psi_{\pi^*\chi}, \tag{29}$$

where $f\in C_Q^\infty(T^*Q)[[\lambda]]$ and $\chi\in C_0^\infty(Q)[[\lambda]]$.

(ii) *The GNS representation π_μ carried to $C_0^\infty(Q)[[\lambda]]$ via Φ is just the Weyl representation ϱ_{W}:*

$$\Phi\pi_\mu(f)\Phi^{-1}\chi = i^*N(f *_{\mathrm{W}} \pi^*\chi) = \varrho_{\mathrm{W}}(f)\chi. \tag{30}$$

Proof. Using the preceding proposition the proof is very simple: First one observes that Φ is indeed well-defined and bijective due to (28) with inverse as stated above. Then

$$\langle\psi_f, \psi_g\rangle = \omega(\overline{f} *_{\mathrm{W}} g) = \int\limits_Q i^*\overline{(Nf)}i^*(Ng)\,\mu$$
$$= \int\limits_Q \overline{\Phi(\psi_f)}\Phi(\psi_g)\,\mu = \langle\Phi(\psi_f), \Phi(\psi_g)\rangle,$$

due to (26) which proves that Φ is isometric. Moreover, we simply calculate

$$\Phi\pi_\mu(f)\Phi^{-1}(\chi) = \Phi\pi_\mu(f)\psi_{\pi^*\chi} = \Phi\psi_{f *_{\mathrm{W}} \pi^*\chi} = i^*N(f *_{\mathrm{W}} \pi^*\chi)$$

and thus (30) directly follows from the definition (6) of ϱ_{W}. □

Notice that in view of Appendix A the whole construction can be done in the setting of formal Laurent or CNP series as well which justifies the terminology of 'Hilbert spaces' and 'Hermitian products' in sense of the definitions in [9, Appendix A, B], in particular $C_0^\infty(Q)\langle\langle\lambda\rangle\rangle$ was shown to be already Cauchy-complete with respect to the topology induced by $\langle\cdot,\cdot\rangle$ in [9, Theorem 7].

5. Several simple physical applications

5.1. Convergence properties

We shall briefly consider the question of convergence if one substitutes the formal parameter λ by the numerical value $\hbar \in \mathbb{R}^+$ of Planck's constant (after having chosen some suitable unit system). In general this is a quite tricky problem and there has been a lot of work (and success!) to understand the convergence properties of the (a priori) formal star products, see e.g. [9,7,11,19,26]. In the situation of homogeneous star products on a cotangent bundle the problem is luckily almost trivial since one has the $\mathbb{C}[\lambda]$-submodule $C_{pp}^\infty(T^*Q)[\lambda]$ of those functions which are polynomial in the momenta. Here the substitution $\lambda \mapsto \hbar$ makes no trouble at all since λ appears only polynomially and furthermore the representations ϱ_{S} and ϱ_{W} make $C_0^\infty(Q)[\lambda]$ to a $C_{pp}^\infty(T^*Q)[\lambda]$-submodule of the $C^\infty(T^*Q)[[\lambda]]$-module $C_0^\infty(Q)[[\lambda]]$ and thus again there is no problem to substitute λ by $\hbar$ in the representations. We shall formulate this obvious fact not as a lemma but mention that the functions polynomial in the momenta are the most interesting observables for the physicist since in many examples the typical Hamiltonians and the typical integrals of motion are of this type.

5.2. Physical examples

Now we want to consider additional structures on T^*Q which are motivated by various physical situations: Usually the Hamiltonian of a free particle moving on Q is determined by a Riemannian metric g on Q by (setting the mass of the particle equal to 1)

$$H_{\mathrm{free}}(q,p) := \tfrac{1}{2}g^\sharp(q)(p,p) = \tfrac{1}{2}g^{ij}(q)p_ip_j, \tag{31}$$

where $g^{\sharp} \in \Gamma(\bigvee^2 TQ)$ is the 'inverse' metric tensor. Obviously $H_{\text{free}} \in C^{\infty}_{pp,2}(T^*Q)$ where the corresponding tensor field is just $g^{\sharp} \in \Gamma(\bigvee^2 TQ)$. In this case we shall use the Levi-Civita connection ∇^{LC} of g to construct the star product $*_{\text{S}}$ and since the metric induces a canonical volume density μ_g we shall use this density to define the one-form α, the corresponding operator N, and the star product of Weyl ordered type $*_{\text{W}}$. Now $\nabla^{\text{LC}}\mu_g = 0$ and hence $\alpha = 0$ which leads to

$$N = \mathrm{e}^{(\lambda/2\mathrm{i})\Delta} \qquad \text{with} \qquad \Delta = \partial_{p_i}\partial_{q^i} + p_r(\pi^*\Gamma^r_{ij})\partial_{p_i}\partial_{p_j} + (\pi^*\Gamma^i_{ij})\partial_{p_j}. \tag{32}$$

Furthermore it is easy to calculate using (7) that the operator corresponding to H_{free} is given by

$$\varrho_{\text{W}}(H_{\text{free}}) = -\frac{\lambda^2}{2}\Delta_g \tag{33}$$

where Δ_g is the Laplacian of the metric g. Moreover *no* 'quantum potentials' in $\varrho_{\text{W}}(H_{\text{free}})$ like e.g. a multiple of the Ricci scalar occur as in the approach of Underhill [29]. If in addition a smooth potential $V \in C^{\infty}(Q)$ is present then the physical Hamiltonian will be given by

$$H = H_{\text{free}} + \pi^*V \tag{34}$$

and the corresponding operator is

$$\varrho_{\text{W}}(H) = -\frac{\lambda^2}{2}\Delta_g + V, \tag{35}$$

where V acts simply as left multiplication. Another important physical example is given by functions linear in the momenta since they generate the point transformations of the configuration space Q. Let $\widehat{X} \in C^{\infty}_{pp,1}(T^*Q)$ be a function linear in the momenta and let $X \in \Gamma(TQ)$ be the corresponding vector field then we obtain

$$\varrho_{\text{W}}(\widehat{X}) = \frac{\lambda}{\mathbf{i}}\left(\mathcal{L}_X + \frac{1}{2}\operatorname{div}_g X\right) \tag{36}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X and $\operatorname{div}_g X$ denotes the metric divergence of X. Note that any homogeneous star product of Weyl type $*$ is covariant under $C^{\infty}_{pp,1}(T^*Q)$, i.e. $\widehat{X} * \widehat{Y} - \widehat{Y} * \widehat{X} = \mathrm{i}\lambda\{\widehat{X}, \widehat{Y}\}$ for all $X, Y \in \Gamma(TQ)$ (see e.g. [2] for further definitions of different types of invariance).

Summarizing we observe that the well-known 'ad hoc quantization rules' on cotangent bundles are obtained by deformation quantization and GNS representation in a very systematic way.

5.3. Symmetries of T^*Q

At last we shall discuss the action of geometrical symmetries of Q and T^*Q as (anti-) automorphisms of the star product algebra $C^{\infty}(T^*Q)[[\lambda]]$. Firstly we shall remember some easy facts about the way (anti-) automorphisms induce (anti-) unitary maps via the GNS construction in a more general setting adapted from the usual C^*-theory (see Appendix A and [9] for definitions, notation and further properties).

Proposition 5.1. *Let $\mathcal{A}$, $\mathcal{B}$ be associative *-algebras over $\mathsf{C} := \mathsf{R}(\mathbf{i})$ where R is an ordered field and $\mathbf{i}^2 = -1$ and * is an involutive C-antilinear (with respect to the complex conjugation in C) anti-automorphism of $\mathcal{A}$ resp. $\mathcal{B}$. Moreover let $\omega : \mathcal{B} \to \mathsf{C}$ be a positive linear functional with Gel'fand ideal $\mathcal{J}_\omega$ and GNS representation π_ω on $\mathfrak{H}_\omega := \mathcal{B}/\mathcal{J}_\omega$. Furthermore let $A : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism and $\tilde{A} : \mathcal{A} \to \mathcal{B}$ be a *-anti-homomorphism. Then we have:*

(i) *The functionals $\omega_A := \omega \circ A$ and $\omega_{\tilde{A}} := \omega \circ \tilde{A}$ are positive linear functionals of $\mathcal{A}$ with Gel'fand ideals $\mathcal{J}_A = A^{-1}(\mathcal{J}_\omega)$ and $\mathcal{J}_{\tilde{A}} = \tilde{A}^{-1}(\mathcal{J}_\omega{}^*)$.*

(ii) *Let π_A resp. $\pi_{\tilde{A}}$ be the GNS representations on $\mathfrak{H}_A := \mathcal{A}/\mathcal{J}_A$ resp. $\mathfrak{H}_{\tilde{A}} := \mathcal{A}/\mathcal{J}_{\tilde{A}}$ induced by ω_A resp. $\omega_{\tilde{A}}$ then the maps*

$$U_A : \begin{array}{c} \mathfrak{H}_A \to \mathfrak{H}_\omega \\ \psi_f^A \mapsto \psi_{Af} \end{array} \quad \textit{resp.} \quad U_{\tilde{A}} : \begin{array}{c} \mathfrak{H}_{\tilde{A}} \to \mathfrak{H}_\omega \\ \psi_f^{\tilde{A}} \mapsto \psi_{\tilde{A}(f^*)} \end{array} \tag{37}$$

(where $\psi_\cdot$, $\psi_\cdot^A$, and $\psi_\cdot^{\tilde{A}}$ denote the equivalence classes in $\mathfrak{H}_\omega$, $\mathfrak{H}_A$, and $\mathfrak{H}_{\tilde{A}}$) are well-defined and isometric resp. anti-isometric and one has

$$U_A\pi_A(f) = \pi_\omega(Af)U_A \quad \textit{resp.} \quad U_{\tilde{A}}\pi_{\tilde{A}}(f) = \pi_\omega(\tilde{A}f^*)U_{\tilde{A}}. \tag{38}$$

(iii) *If moreover A resp. $\tilde{A}$ is surjective then U_A resp. $U_{\tilde{A}}$ is unitary resp. anti-unitary and the inverse of U_A resp. $U_{\tilde{A}}$ is (well-defined!) given by*

$$U_A^{-1}\psi_g = \psi_f^A \quad \textit{resp.} \quad U_{\tilde{A}}^{-1}\psi_g = \psi_f^{\tilde{A}} \tag{39}$$

where $f \in A^{-1}(\{g\})$ resp. $f \in \tilde{A}^{-1}(\{g^*\})$.

Proof. The proof is a straightforward computation using the (anti-) homomorphism property of A (resp. $\tilde{A}$) as well as the compatibility of A and $\tilde{A}$ with the *-involution. □

A homomorphism which respects the involution will also be called a real homomorphism.

Corollary 5.2. *With the notations from above let A (resp. $\tilde{A}$) : $\mathcal{A} \to \mathcal{A}$ be a real (anti-) automorphism and let $\omega : \mathcal{A} \to \mathsf{C}$ be a positive A-invariant (resp. $\tilde{A}$-invariant) linear functional, i.e. $\omega = \omega \circ A$ (resp. $\omega = \omega \circ \tilde{A}$). Then U_A (resp. $U_{\tilde{A}}$) : $\mathfrak{H}_\omega \to \mathfrak{H}_\omega$ is a (anti-) unitary map with inverse $U_A^{-1}\psi_f = \psi_{A^{-1}f}$ resp. $U_{\tilde{A}}^{-1}\psi_f = \psi_{\tilde{A}^{-1}f^*}$ and*

$$\pi_\omega(f) = U_A^{-1}\pi_\omega(Af)U_A \quad \textit{resp.} \quad \pi_\omega(f) = U_{\tilde{A}}^{-1}\pi_\omega(\tilde{A}f^*)U_{\tilde{A}}. \tag{40}$$

Note that all these results are still correct if one considers an ordered ring R instead of an ordered field as we shall do in the following using $\mathbb{R}[[\lambda]]$ and $\mathbb{C}[[\lambda]]$.

Now we come back to the particular situation T^*Q. Let $\phi : Q \to Q$ be a diffeomorphism of Q leaving invariant the connection ∇, i.e. $\nabla_X Y = \phi^*(\nabla_{\phi_* X}\phi_* Y)$ for all vector fields $X, Y \in \Gamma(TQ)$. Moreover, we consider the canonical lift $T^*\phi : T^*Q \to T^*Q$ of ϕ to a symplectic diffeomorphism of T^*Q (where we use the convention such that $T^*\phi \circ i = i \circ \phi$ and not $i \circ \phi^{-1}$).

Lemma 5.3. *Let ∇ be a ϕ-invariant torsion-free connection on Q where $\phi : Q \to Q$ is a diffeomorphism then the pull back $A_\phi := (T^*\phi)^* : C^\infty(T^*Q)[[\lambda]] \to C^\infty(T^*Q)[[\lambda]]$ is a real automorphism of the corresponding star product $*_{\mathrm{S}}$, i.e.*

$$A_\phi(f *_{\mathrm{S}} g) = A_\phi f *_{\mathrm{S}} A_\phi g. \tag{41}$$

Proof. We shall only indicate the proof very briefly: First we notice that in the Fedosov setting as desribed in [8] the map A_ϕ naturally extends to the whole Fedosov algebra and defines a real automorphism of the fibrewise standard ordered product. This is a straightforward computation using the invariance of ∇ under ϕ. Next one proves that A_ϕ commutes with the Fedosov derivation $\mathcal{D}_{\mathrm{S}}$ using the recursion formulas for the curvature part of $\mathcal{D}_{\mathrm{S}}$. Finally one applies Proposition 2.3 in [8] and the rather obvious fact that A_ϕ commutes with the Fedosov Taylor series to prove the fact that the fibrewise automorphism induces indeed an automorphism of the star product, namely A_ϕ. Another possibility is given by proving (41) by first restricting to functions polynomial in the momenta which turns out to be sufficient to prove (41) and secondly using the explicit formula for the representation ϱ_{S} and the invariance of ∇. □

Now if A_ϕ is an automorphism of $*_{\mathrm{S}}$ it is an automorphism of $*_{\mathrm{W}}$ too, if it commutes with the operator N which is clearly the case if the one-form α is ϕ-invariant: $\phi^*\alpha = \alpha$. Note that a priori we only know from the ϕ-invariance of ∇ that $\mathrm{d}\alpha = -\mathrm{tr} R$ is ϕ-invariant. But assuming $\phi^*\alpha = \alpha$ we obtain that A_ϕ is an automorphism of $*_{\mathrm{W}}$ as well. Note that in the case where ∇ is uni-modular, i.e. $\mathrm{tr} R = 0$ (and in particular in the Riemannian case with the Levi-Civita connection), with the choice $\alpha = 0$ this will automatically be fulfilled.

Corollary 5.4. *Let $\alpha \in \Gamma(T^*Q)$ be a one-form such that $\mathrm{d}\alpha = -\mathrm{tr} R$ and $\phi^*\alpha = \alpha$ then $A_\phi = (T^*\phi)^*$ is a real automorphism of the corresponding star product $*_{\mathrm{W}}$ and commutes with N.*

Assuming in addition that we have a ϕ-invariant volume density μ we automatically obtain that α defined by (20) is ϕ-invariant. Moreover, ω_μ defined as in (24) is an A_ϕ-invariant positive linear functional and hence we can apply Corollary 5.2 to obtain the following easy lemma:

Lemma 5.5. *Let both ∇ and μ be ϕ-invariant then $\phi^*\alpha = \alpha$ and A_ϕ is a real $*_{\mathrm{W}}$-automorphism. Furthermore ω_μ is A_ϕ-invariant and the induced unitary map U_ϕ in the GNS Hilbert space is simply given by*

$$U_\phi \chi = \phi^* \chi, \qquad \chi \in C_0^\infty(Q)[[\lambda]]. \tag{42}$$

This lemma covers obviously all kinds of (Lie-) group actions on Q which leave invariant a connection and a volume density and ensures that such classical group actions are implemented as group actions of automorphisms of the observable algebra as well as unitray group actions on the GNS Hilbert space both realized by pull backs. This is of course a well-known implementation but again deformation quantization and the GNS representation provide a very systematical way to study such geometric symmetries and their realizations in quantum mechanics.

At last we shall discuss the 'time reversal' on a kinematical level, i.e. as a geometrical property of the classical phase space (see e.g. [1, p. 308]):

Definition 5.6. Let (M, ω) be a symplectic manifold and $T : M \to M$ a diffeomorphism then T is called time reversal map if $T^*\omega = -\omega$.

Lemma 5.7. *If (M, ω) has a time reversal map T then $T \circ \phi$ is again a time reversal map for any symplectic diffeomorphism ϕ and any time reversal map of M is obtained this way. If ω is not exact then T is a 'large diffeomorphism' i.e. not isotopic to the identity. If the fixed point set of T is a submanifold then it is isotropic.*

In the case of a cotangent bundle we canonically have such a time reversal map namely $T : T_q^*Q \ni \alpha_q \mapsto -\alpha_q$ with the additional property $T^2 = \mathsf{id}$. In a local bundle Darboux chart this reads $T : (q, p) \mapsto (q, -p)$ which justifies the name 'time reversal'. The following proposition should be well known:

Proposition 5.8. *Let (M, ω) be a symplectic manifold and $T : M \to M$ a time reversal map with $T^2 = \mathsf{id}_M$. Then the set of fixed points of T is either empty or a (not necessarily connected) Lagrangian submanifold of M.*

Proof. First we notice that there always exists a symplectic torsion-free and T-invariant connection ∇. Now assume that $L := \{p \in M | T(p) = p\}$ is not empty and let $p \in L$. Then $T_pT : T_pM \to T_pM$ has square id_{T_pM} and hence it is diagonalizable with eigenspaces $E_p^{\pm}$ to the eigenvalues ± 1. Clearly $E_p^{\pm}$ are both Lagrangian subvector spaces of T_pM. Now the exponential mapping $\exp_p$ of ∇ is locally a diffeomorphism and maps a neigbourhood of 0_p in E_p^+ into L due to the T-invariance of ∇ and thus the local inverse of $\exp_p$ is a submanifold chart for L in a neighbourhood of p proving thereby that L is indeed a submanifold (even totally geodesic with respect to ∇) with tangent space E_p at p for all $p \in L$. This implies that L is Lagrangian. □

Certain coadjoint orbits carry time reversal maps with nonempty fixed-point set:

Proposition 5.9. *Let $\mathfrak{g}$ a real finite-dimensional Lie algebra and $\mathfrak{g}^*$ its dual space. For any $\mu \in \mathfrak{g}^*$ consider the coadjoint orbit $\mathcal{O}_\mu$ through μ, i.e. $\mathcal{O}_\mu := \{\exp(\mathrm{ad}^*(\xi_1)) \cdots \exp(\mathrm{ad}^*(\xi_n))\mu \in \mathfrak{g}^* \mid n \in \mathbb{N}, \xi_1, \ldots, \xi_n \in \mathfrak{g}\}$.*

(i) *Let furthermore $s : \mathfrak{g} \to \mathfrak{g}$ an involutive automorphism of Lie algebras (i.e. $s^2 = \mathsf{id}_{\mathfrak{g}}$). Then $\mathfrak{g}^*$ decomposes into the direct sum of the eigenspaces $\mathfrak{k}^*$ and $\mathfrak{m}^*$ of s^*, the dual map to s, corresponding to the eigenvalues 1 and -1, respectively. Pick $\mu \in \mathfrak{m}^*$ and consider the coadjoint orbit $\mathcal{O}_\mu$ through μ. Then the map $T : \mathcal{O}_\mu \to \mathcal{O}_\mu$ defined by the restriction of $-s^*$ is an involutive time reversal map of the symplectic manifold $\mathcal{O}_\mu$*

with its Kirillov–Kostant–Souriau form whose fixed point set is equal to the intersection of $\mathfrak{m}^*$ *and* $\mathcal{O}_\mu$.

(ii) *For each element* μ *in the dual space* $\mathfrak{g}^*$ *of a semisimple compact Lie algebra* $\mathfrak{g}$ *there is an involutive automorphism* s *of* $\mathfrak{g}$ *such that* μ *is in the eigenspace* $\mathfrak{m}^*$ *of* s^*. *In view of (i) it follows that every coadjoint orbit in* $\mathfrak{g}^*$ *with semisimple compact* $\mathfrak{g}$ *admits an involutive time reversal map with nonempty fixed point set.*

Proof.

(i) Since $s^*\mathrm{ad}^*(\xi) = \mathrm{ad}^*(s\xi)s^*$ it follows that $-s^*$ restricts to $\mathcal{O}_\mu$ whence T is well-defined and the fixed point set is equal to the intersection of the orbit with $\mathfrak{m}^*$. Moreover, this implies that for all $\xi, \eta \in \mathfrak{g}$ and $\nu \in \mathcal{O}_\mu$ we have $T_\nu T\xi_{\mathcal{O}_\mu}(\nu) = (s\xi)_{\mathcal{O}_\mu}(T\nu)$ for all the vector fields $\xi_{\mathcal{O}_\mu}(\nu) = \mathrm{ad}^*(\xi)\nu$ implying that $(T^*\omega)(\nu)(\xi_{\mathcal{O}_\mu}(\nu), \eta_{\mathcal{O}_\mu}(\nu)) = -(s^*\nu)([s\xi, s\eta]) = -\nu([\xi,\eta]) = -\omega(\nu)(\xi_{\mathcal{O}_\mu}(\nu), \eta_{\mathcal{O}_\mu}(\nu))$.

(ii) Identifying the Lie algebra and its dual space by means of the Killing form we can put $\mu \in \mathfrak{g}$ in some Cartan subalgebra $\mathbf{i}\mathfrak{h}$ of $\mathfrak{g}$. Using the root space decomposition

$$\mathfrak{g} = \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \mathbf{i}\mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathbb{R}\mathbf{i}(X_\alpha + X_{-\alpha}),$$

where Δ is the set of all roots of $\mathfrak{g}$, and X_α are normalized vectors in the eigenspaces $\mathfrak{g}_\alpha$ (see [21, p. 182] for definitions and details) we see that the first summand is a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ whereas the second plus the third summand is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$, where we have made use of [21, Theorem 5.5]. Hence $\mathfrak{g}$ is an orthogonal symmetric Lie algebra (see [21, p. 213]) where s can be defined by being 1 on $\mathfrak{k}$ and -1 on $\mathfrak{m}$. □

Now we come back to cotangent bundles and consider a Fedosov star product of Weyl ordered type $*_{\mathrm{W}}$ on T^*Q then we have the following lemma which is proved directly using the Weyl type property and the homogeneity of $*_{\mathrm{W}}$:

Lemma 5.10. *The canonical time reversal map* T *induces via pull back a real anti-automorphism of any* $*_{\mathrm{W}}$ *on* T^*Q, *i.e. for any* $f, g \in C^\infty(T^*Q)[[\lambda]]$ *and* $A_T := T^*$ *we have*

$$A_T(f *_{\mathrm{W}} g) = A_T g *_{\mathrm{W}} A_T f. \tag{43}$$

Since A_T is an anti-automorphism of $*_{\mathrm{W}}$ we obtain an anti-unitary map U_T in each GNS Hilbert space $\mathfrak{H}_\mu$ since obviously $\omega_\mu \circ A_T = \omega_\mu$ for all volume densities μ. As expected it turns out that U_T is just the complex conjugation of the wave functions:

Lemma 5.11. *Let* A_T *be the time reversal* $*_{\mathrm{W}}$*-anti-automorphism and let* ω_μ *be the positive linear functional as in* (24) *then the induced anti-unitary map* U_T *in the GNS Hilbert space* $C_0^\infty(Q)[[\lambda]]$ *is given by complex conjugation*

$$U_T\chi = \overline{\chi}, \qquad \chi \in C_0^\infty(Q)[[\lambda]]. \tag{44}$$

6. Comparison with other approaches and integral formulas

In this section we shall compare our differential operator representations we constructed to some different approaches using integral representations. Before we can translate our explicit formulas to integral expressions we have to substitute the formal parameter λ by the real positive number $\hbar \in \mathbb{R}^+$ which is done most easily by restricting to functions polynomial in the momenta. The substitution $\lambda \mapsto \hbar \in \mathbb{R}^+$ is denoted by $\ldots|_{\lambda=\hbar}$. In general the integral expressions will be well defined for a larger class of functions namely certain symbol classes which are smooth functions on T^*Q with controllable fibrewise increase. Then our formal results can be obtained by asymptotic expansions.

First of all we shall show that the integral expression given by [26] in the framework of symbol calculus when applied to functions that are polynomial in the momenta coincides with the representation ϱ_S we introduced in a purely algebraic fashion. Moreover, we define some integral representations that are quite natural generalizations of the Weyl quantization in the case of flat $T^*\mathbb{R}^n$ (cf. [19]) and have been already studied by [16,17] in the case of a Riemannian manifold Q. In so far our results are generalizing since we drop those technically simplifying preconditions. On the other hand we can show that this alternative approach to a Weyl quantization directly yields (when applied to polynomial functions in the momenta) the Weyl representation we used to define our star product $*_W$ in order to do the GNS construction.

6.1. An integral expression for the standard representation

In the flat case it is well known that the standard representation $\varrho_S(\widehat{T})$ of $\widehat{T} \in C^\infty_{pp}(T^*\mathbb{R}^n)$ applied to a function $\phi \in C^\infty_0(\mathbb{R}^n)$ can be given by the integral representation

$$(\varrho_S(\widehat{T})\phi)(q)|_{\lambda=\hbar} = \frac{1}{(2\pi\hbar)^n} \int\limits_{\mathbb{R}^n} \widehat{T}(q,p) \int\limits_{\mathbb{R}^n} \mathrm{e}^{-(\mathrm{i}/\hbar)\langle p,v\rangle} \phi(q+v)\,\mathrm{d}^n v\,\mathrm{d}^n p,$$

and therefore we first report on the generalization of such an expression as it has been proposed in [26] and give an analogue that can be defined pointwise for a fixed $q \in Q$ by integrations over $T^*_q Q$ and $T_q Q$ using the canonical symplectic volume form Ω_q on $T_q Q \times T^*_q Q$. The expression $\phi(q+v)$ in the above formula obviously makes no sense in the case of an arbitrary manifold Q but can be viewed as $\phi(\exp_q(v_q))$ for $v_q \in T_q\mathbb{R}^n$ denoting by exp the exponential mapping with respect to the flat connection. In general the exponential mapping not being globally defined we aim to define such an expression just in a neighbourhood of 0_q. To do so, we choose an open neighbourhood $\mathcal{O} \subseteq TQ$ of the zero section that is mapped diffeomorphically to an open neighbourhood of the diagonal in $Q \times Q$ by $\pi \times \exp$. By possibly shrinking this neighbourhood we may assume that for every $v_q \in \pi^{-1}(q) \cap \mathcal{O}$ also $-v_q$ is contained in $\pi^{-1}(q) \cap \mathcal{O}$. This will be important for the definition of the Weyl representation later. By Urysohn's lemma there is a smooth function $\chi : TQ \to [0,1]$ and an open neighbourhood $\widetilde{\mathcal{O}} \subseteq \mathcal{O}$ of the zero section such that $\chi|_{\widetilde{\mathcal{O}}} = 1$ and $\mathrm{supp}(\chi) \subseteq \mathcal{O}$. Using this so-called cut-off function we may assign to any smooth function ϕ on Q a function $\phi_q^\chi \in C_0^\infty(T_q Q)$ by

$$\phi_q^\chi(v_q) := \begin{cases} \chi(v_q)\phi(\exp_q(v_q)) & \text{for } v_q \in \pi^{-1}(q) \cap \mathcal{O}, \\ 0 & \text{else.} \end{cases}$$

After these preparations the mapping

$$\mathcal{S}(\widehat{T}) : C^\infty(Q) \ni \phi \mapsto \left(q \mapsto \frac{1}{(2\pi\hbar)^n} \int\limits_{T_q^* Q} \widehat{T}(\alpha_q) \int\limits_{T_q Q} \mathrm{e}^{-(\mathrm{i}/\hbar)\alpha_q(v_q)} \phi_q^\chi(v_q) \Omega_q \right) \tag{45}$$

is obviously well-defined for $\widehat{T} \in C_{pp}^\infty(T^*Q)$ by well-known properties of the Fourier transform. At first sight it might seem that this expression depends on the special choice of the function χ, what actually is not the case for functions polynomial in the momenta, and in addition it is a priori not clear whether the function $\mathcal{S}(\widehat{T})\phi$ is again a smooth function. But we have the following:

Lemma 6.1. *For all $\widehat{T} \in C_{pp}^\infty(T^*Q)$ and $\phi \in C^\infty(Q)$ we have*

$$\mathcal{S}(\widehat{T})\phi = \varrho_{\mathrm{S}}(\widehat{T})\phi|_{\lambda=\hbar}.$$

Proof. First of all we notice that $(\mathcal{S}(\widehat{T})\phi)(q)$ is defined independently on the choice of the basis of $T_q Q$ and $T_q^* Q$ one introduces to carry out the integrations, therefore it is convenient to use a coordinate basis induced by a normal chart around q. In addition it suffices to prove the assertion for homogeneous $\widehat{T}$. By an obvious calculation one gets $(\mathcal{S}(\widehat{T})\phi)(q) = (\hbar/\mathrm{i})^k (1/k!) T^{i_1 \cdots i_k}(q) \partial^k \phi_q^\chi(v_q)/(\partial v^{i_1} \cdots \partial v^{i_k})|_{v_q = 0_q}$. Now by construction of the cut-off function χ all its derivatives at 0_q vanish and we get the coordinate expression for $(\varrho_{\mathrm{S}}(\widehat{T})\phi)(q)|_{\lambda=\hbar}$ in a normal chart around q proving the assertion since in a normal chart around q the symmetric covariant derivative is given by $(D^k\phi)(q) = (\partial^k \phi/\partial q^{i_1} \cdots \partial q^{i_k}(q))\, \mathrm{d}q^{i_1} \vee \cdots \vee \mathrm{d}q^{i_k}$ (cf. [8, Lemma A.9]). □

6.2. An integral expression for the Weyl representation

Again motivated by the flat case in [16,17] an integral expression has been introduced to generalize the Weyl quantization in the following manner. For $\phi, \psi \in C_0^\infty(Q)$ and $\widehat{T} \in C_{pp}^\infty(T^*Q)$ one considers using the same notation as in Section 6.1 the mapping

$$\mathsf{W}(\widehat{T}) : (\phi, \psi) \mapsto \left(q \mapsto \frac{1}{(\pi\hbar)^n} \int\limits_{T_q^* Q} \widehat{T}(\alpha_q) \right.$$
$$\left. \times \int\limits_{T_q Q} \mathrm{e}^{-(2\mathrm{i}/\hbar)\alpha_q(v_q)} \overline{\phi_q^\chi(-v_q)} \psi_q^\chi(v_q) \Omega_q \right),$$

that shall serve as an integral kernel to define the Weyl representation $\mathcal{W}(\widehat{T})\psi$ by its 'Matrix elements' $\langle \phi, \mathcal{W}(\widehat{T})\psi\rangle := \int_Q \mathsf{W}(\widehat{T})(\phi, \psi)\,\mu$. This is clearly well-defined, since $(C_0^\infty(Q), \langle\,\cdot\,,\,\cdot\,\rangle)$ is a pre-Hilbert space and as we shall see in the next lemma $\mathsf{W}(\widehat{T})$ is a differential operator acting on the functions ϕ, ψ.

Lemma 6.2. *$\mathsf{W}(\widehat{T})(\phi, \psi)$ is defined independently on the choice of the cut-off function χ and in a local chart of Q it is given for $\widehat{T} \in C^\infty_{pp,k}(T^*Q)$ by*

$$\mathsf{W}(\widehat{T})(\phi, \psi) = \frac{1}{k!}\left(\frac{\hbar}{2\mathrm{i}}\right)^k i_s^*(\mathrm{d}q^{i_1})\dots i_s^*(\mathrm{d}q^{i_k})T$$
$$\times \sum_{r=0}^{k}(-1)^r\binom{k}{r} i_s(\partial_{q^{i_1}})\dots i_s(\partial_{q^{i_r}})\frac{1}{r!}D^r\overline{\phi}\, i_s(\partial_{q^{i_{r+1}}})\dots i_s(\partial_{q^{i_k}})\frac{1}{(k-r)!}D^{k-r}\psi,$$

hence it obviously is smooth, and since the functions ϕ, ψ have compact support this implies that $\int_Q \mathsf{W}(\widehat{T})(\phi, \psi)\mu$ is well-defined.

Proof. By a straightforward computation as in the case of the standard representation using a normal chart around q one obtains $\mathsf{W}(\widehat{T})(\phi, \psi)(q) = (1/k!)(\hbar/2\mathrm{i})^k T^{i_1\cdots i_k}(q)(\partial^k(\overline{\phi_q^\chi(-v_q)}\,\psi_q^\chi(v_q))/\partial v^{i_1}\cdots\partial v^{i_k})|_{v_q=0_q}$. By the Leibniz rule and the construction of χ one gets the assertion observing the shape of the symmetric covariant derivative in a normal chart around q. □

After this preparation we can state the following lemma:

Lemma 6.3. *The Weyl representation $\mathcal{W}(\widehat{T})\psi$ being well defined by the equation $\langle\phi, \mathcal{W}(\widehat{T})\psi\rangle = \int_Q \mathsf{W}(\widehat{T})(\phi, \psi)\,\mu$ coincides with the Weyl-representation ϱ_{W} defined as in* (6) *after the substitution $\lambda \mapsto \hbar$, i.e.*

$$\mathcal{W}(\widehat{T})\psi = \varrho_{\mathrm{W}}(\widehat{T})\psi|_{\lambda=\hbar} = \varrho_{\mathrm{S}}(N\widehat{T})\psi|_{\lambda=\hbar},$$

where N is given as in (2).

Proof. The proof is an easy but lengthy calculation using iterated integrations by parts in the same fashion as they were nessecary to prove Eqs. (23). □

Finally, note that other approaches like Underhill's (see [29]) use compactly supported half-densities as representation space instead of compactly supported functions on Q. But although there is an isomorphism between these spaces by choosing a fixed positive density μ and assigning to $\phi \in C_0^\infty(Q)$ the half-density $\check{\phi} = \phi\mu^{1/2}$ the corresponding operators turn out to differ e.g. by additional terms proportional to $\hbar^2$ times the scalar curvature for the free particle Hamiltonian (31): this difference is due to whether the reference density $\mu^{1/2}$ is pulled-back by means of the exponential map or not, see e.g. [16, pp. 518–520] for a detailed discussion.

7. WKB expansion for projectable Lagrangian submanifolds

In this section we shall discuss how the usual WKB expansion can be formulated in the framework of deformation quantization using particular GNS representations in the case of cotangent bundles. We consider here a Hamiltonian $H \in C^\infty(T^*Q)$ and assume that for a fixed energy value $E \in \mathbb{R}$ in the image of H there exists a Lagrangian submanifold L_β contained in $H^{-1}(\{E\})$, i.e.

$$H|L_\beta = E, \tag{46}$$

which is furthermore given by the graph of a closed one-form $\beta \in \Gamma(T^*Q)$, i.e.

$$L_\beta = \mathrm{graph}(\beta) = \{\alpha_q \in T^*Q \mid \forall q \in Q : \alpha_q = \beta(q)\}. \tag{47}$$

Such Lagrangian submanifolds are called projectable and it is well known that $\mathrm{d}\beta = 0$ is equivalent to the statement that graph(β) is Lagrangian, see e.g. [1, Prop. 5.3.15]. For later use we denote by

$$i_\beta : L_\beta \to T^*Q, \tag{48}$$

the embedding of L_β in T^*Q. Since $\mathrm{d}\beta = 0$ we have $\mathrm{d}\pi^*\beta = 0$ which implies that the vector field $X \in \Gamma(T(T^*Q))$ defined by $i_X\omega_0 = \pi^*\beta$ is symplectic with the following well-known properties (see e.g. [3, Sec. 3.2]):

Lemma 7.1. *Let $\beta \in \Gamma(T^*Q)$ be a closed one-form then the symplectic flow ϕ_s of the symplectic vector field X defined by $i_X\omega_0 = \pi^*\beta$ is complete and given by*

$$\phi_s(\alpha_q) = \alpha_q - s\beta(q) \tag{49}$$

for $\alpha_q \in T^*_qQ$ and $s \in \mathbb{R}$ and for $L_{s\beta} :=$ graph$(s\beta)$ we have

$$\phi_{-s}(i(Q)) = i_{s\beta}(L_{s\beta}). \tag{50}$$

Clearly ϕ_{-s} determines a diffeomorphism $\Phi_s : Q \to L_{s\beta}$ by $q \mapsto s\beta(q)$ such that

$$\phi_{-s} \circ i = i_{s\beta} \circ \Phi_s \tag{51}$$

for all $s \in \mathbb{R}$. For $s = 1$ we obtain from (46) and (50)

$$i^*\phi^*_{-1}H = E. \tag{52}$$

Until now the setting was completely classical and we shall now construct a quantum mechanical automorphism as analogue to ϕ_s using Appendix B: Since $\mathrm{d}\pi^*\beta = 0$ the quantum mechanical time evolution with respect to this one-form determines a one-parameter group of real automorphisms $A_s : C^\infty(T^*Q)[[\lambda]] \to C^\infty(T^*Q)[[\lambda]]$ of the star product $*_\mathrm{W}$ of the form

$$A_s = \phi_s^* \circ T_s, \qquad \text{where} \qquad T_s = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_s^{(r)} \tag{53}$$

and each $T_s^{(r)}$ is a differential operator which can be computed in principle by iterated integrals as in Appendix B. Note that $A_s\pi^* = \pi^*$. Then the image of the two-sided ideal $C_Q^\infty(T^*Q)[[\lambda]]$ under A_s is again a two-sided ideal stable under complex conjugation since A_s is real and it can easily be determined: let $C_{L_{s\beta}}^\infty(T^*Q)$ be the set of those functions f such that $\operatorname{supp} f \cap i_{s\beta}(L_{s\beta})$ is compact, then we have the following lemma:

Lemma 7.2. *With the notations from above we have for all $s \in \mathbb{R}$*

$$A_s(C_Q^\infty(T^*Q)[[\lambda]]) = C_{L_{s\beta}}^\infty(T^*Q)[[\lambda]] \tag{54}$$

and hence the $\mathbb{C}[[\lambda]]$-linear functional

$$\omega_s := \omega_\mu \circ A_{-s} : \; C_{L_{s\beta}}^\infty(T^*Q)[[\lambda]] \;\to\; \mathbb{C}[[\lambda]] \tag{55}$$

is well-defined and positive and for $f \in C_{L_{s\beta}}^\infty(T^*Q)[[\lambda]]$ we have

$$\omega_s(f) = \int\limits_{L_{s\beta}} i_{s\beta}^* \left(T_{-s}(f)\right) \; \mu_s, \qquad \textit{where } \mu_s := {\Phi_s^{-1}}^* \mu. \tag{56}$$

Proof. Eq. (54) follows easily from (53) and (50) since $T_s^{(r)}$ is a differential operator. Then the well-definedness of ω_s is obvious and the positivity follows from the fact that A_{-s} is a real $*_{\mathrm{W}}$-automorphism. Eq. (56) is a straightforward computation using (51) and (53). □

Now we can apply Proposition 5.1 and consider the GNS representation π_s induced by ω_s on the Hilbert space $\mathfrak{H}_s := C_{L_{s\beta}}^\infty(T^*Q)[[\lambda]]/\mathcal{J}_s$ where $\mathcal{J}_s$ is the Gel'fand ideal of ω_s.

Lemma 7.3. *The Gel'fand ideal $\mathcal{J}_s$ of ω_s is given by $\mathcal{J}_s = A_s(\mathcal{J}_\mu)$ and $\mathfrak{H}_s$ is canonically isometric to $C_0^\infty(L_{s\beta})[[\lambda]]$ endowed with the Hermitian product*

$$\langle \chi, \varphi \rangle_s := \int\limits_{L_{s\beta}} \overline{\chi}\varphi \; \mu_s, \qquad \text{where } \chi, \varphi \in C_0^\infty(L_{s\beta})[[\lambda]] \tag{57}$$

via the $\mathbb{C}[[\lambda]]$-linear unitary map

$$\psi_f^{(s)} \;\mapsto\; {\Phi_s^{-1}}^* i^* N A_{-s}(f) \quad \textit{and its inverse} \quad \chi \;\mapsto\; \phi_{\pi^*\Phi_s^*\chi}^{(s)}, \tag{58}$$

where $f \in C_{L_{s\beta}}^\infty(T^*Q)[[\lambda]]$ and $\chi \in C_0^\infty(L_{s\beta})[[\lambda]]$. Moreover, A_s induces due to Proposition 5.1 a unitary map $U_s : C_0^\infty(Q)[[\lambda]] \to C_0^\infty(L_{s\beta})[[\lambda]]$ which is given by

$$U_s : \; \chi \mapsto {\Phi_s^{-1}}^* \chi \qquad \text{for } \chi \in C_0^\infty(Q)[[\lambda]] \tag{59}$$

and the induced GNS representation π_s on $C_0^\infty(L_{s\beta})[[\lambda]]$ is given by

$$\pi_s(f)\chi = U_s \varrho_{\mathrm{W}}(A_{-s}f) U_s^{-1}\chi = {\Phi_s^{-1}}^* \varrho_{\mathrm{W}}(A_{-s}f)\Phi_s^*\chi \tag{60}$$

for all $f \in C^\infty(T^*Q)[[\lambda]]$ and $\chi \in C_0^\infty(L_{s\beta})[[\lambda]]$.

Proof. It is a straightforward computation to prove that the first map in (58) is well-defined, bijective and has the inverse as in (58) due to $A_s\pi^* = \pi^* = N\pi^*$. The fact that (58) is isometric is computed the same way. Using the definition of the unitary map as in Proposition 5.1 and (58) Eqs. (59) and (60) easily follow. □

The WKB expansion is now obtained from the following eigenvalue problem: We consider the case $s = 1$ then $L_\beta \subseteq H^{-1}(\{E\})$ and (52) is valid. Then we ask for an eigenfunction $\tilde{\chi} \in C_0^\infty(L_\beta)[[\lambda]]$ of $\pi_1(H)$ with eigenvalue E, i.e.

$$\pi_1(H)\tilde{\chi} = E\tilde{\chi} \tag{61}$$

and due to (60) it is equivalent to the corresponding eigenvalue problem in $C_0^\infty(Q)[[\lambda]]$ namely

$$\varrho_{\mathrm{W}}(A_{-1}H)\chi = E\chi, \tag{62}$$

where $\chi = \Phi_1^*\tilde{\chi}$. Now (62) is in each order of λ a coupled linear partial differential equation for $\chi_r \in C_0^\infty(Q)$ if we write $\chi = \sum_{r=0}^\infty \lambda^r \chi_r$. Hence it is well defined to ask for a solution in a distributional sense, i.e. we ask for $\chi \in C_0^\infty(Q)'[[\lambda]]$ to solve (62) where $C_0^\infty(Q)'$ denotes the space of distributions on Q which is obtained as usual as the topological dual of $C_0^\infty(Q)$ with respect to its locally convex topology. The main observation is now that we get a linear (first order) partial differential equation for the χ_r which can be solved recursively. The result is a straightforward computation expanding (62) in powers of λ completely analogously to [10, Theorem 3] (where we used a slightly different notation):

Theorem 7.4 (Formal WKB expansion). *Let $H \in C^\infty(T^*Q)$ and $L_\beta \subseteq H^{-1}(\{E\})$ be a projectable Lagrangian submanifold such that $L_\beta = \mathrm{graph}(\beta)$ with $\beta \in \Gamma(T^*Q)$ and let A_s be the one-parameter group of $*_{\mathrm{W}}$-automorphisms induced by β. Then the WKB eigenvalue problem* (62) *for $\chi = \sum_{r=0}^\infty \lambda^r \chi_r \in C_0^\infty(Q)'[[\lambda]]$ is equivalent to the following recursive system of linear first order partial differential equations for χ:*
The homogeneous WKB transport equation for χ_0

$$i^*\{\mathrm{i}\phi_{-1}^*H, \pi^*\chi_0\} + i^*\left(\frac{\mathrm{i}\Delta}{2}\phi_{-1}^*H + \phi_{-1}^*T_{-1}^{(1)}H\right)\chi_0 = 0 \tag{63}$$

and the inhomogeneous WKB transport equation for χ_r, $r \geq 1$

$$\begin{aligned} &i^*\left\{\mathrm{i}\phi_{-1}^*H, \pi^*\chi_r\right\} + i^*\left(\frac{\mathrm{i}\Delta}{2}\phi_{-1}^*H + \phi_{-1}^*T_{-1}^{(1)}H\right)\chi_r \\ &= -\sum_{\substack{a+b+c+d=r+1\\ a,b,c,d\geq 0, d<r}} i^*M_a\left(\frac{1}{b!}\left(\frac{\mathrm{i}\Delta}{2}\right)^b \phi_{-1}^*T_{-1}^{(c)}H, \pi^*\chi_d\right), \end{aligned} \tag{64}$$

where Δ as in (1) *and M_a is the following bidifferential operator as in the standard ordered representation, i.e.*

$$i^* M_a(f, \pi^*\chi) = \frac{1}{a!\mathrm{i}^a} i^* \left(\frac{\partial^a f}{\partial p_{i_1} \cdots \partial p_{i_a}} \right) i_s(\partial_{q^{i_1}}) \cdots i_s(\partial_{q^{i_a}}) \frac{1}{a!} D^a \chi \tag{65}$$

and $T_{-1}^{(c)}$ *as in* (53).

Proof. The proof is a simple computation of $\varrho_{\mathrm{W}}(A_{-1}H)\chi$ by expanding this in powers of λ and (52). □

Note that the question whether this system has a formal solution χ and whether this solution can be 'summed up' to an eventually regular distribution after the substitution $\lambda \to \hbar$ is not answered by this theorem. Nevertheless it provides a rather explicit recursion scheme for the formal eigendistribution χ (the only problem are the operators $T_{-1}^{(c)}$ which are less explicit but could be computed by iterated integrals as in Appendix B). In physical applications the Hamiltonian is often of the form (34) and in this case the above recursion scheme is simplified even more due to the following lemma which implies that in the WKB transport equations all terms involving the operators $T_{-1}^{(c)}$ with $c \geq 1$ applied to such Hamiltonians vanish and thus all operators in the recursion are explicitly given.

Lemma 7.5. *Let* $H \in C^\infty_{pp}(T^*Q)$ *be at most quadratic in the momenta and let* $\beta \in \Gamma(T^*Q)$ *be a closed one-form and* $A_s = \phi_s^* \circ T_s$ *the corresponding time development operator. Then* $A_s H = \phi_s^* H$ *and thus* $T_s H = H$.

The proof of this lemma is a simple consequence of the homogeneity and the Weyl type property of $*_{\mathrm{W}}$.

Note furthermore that the usual WKB phase can be recovered by the same argument as in [10] from the $*_{\mathrm{W}}$-automorphism A_s interpreted as the conjugation by the *star-exponential* $\mathrm{e}^{*_{\mathrm{W}}\mathrm{i}\pi^* S/\hbar} = \mathrm{e}^{\mathrm{i}\pi^* S/\hbar}$ (see [4] for a definition) where locally S is a solution of the Hamilton Jacobi equation, i.e. $\mathrm{d}S = \beta$ and λ is substituted by $\hbar$.

8. The trace for homogeneous star products on T^*Q

This section was motivated by a conversation with Markus Pflaum who proved a particular case of the result we present in this section by using symbol calculus for pseudo-differential operators on Riemannian manifolds (cf. [27]). Compare also with [12] for the case of a compact Riemannian configuration space where another calculus for pseudo-differential operators was used.

We shall show in this section in a more algebraical way that the integration over T^*Q is a trace (i.e. a linear form vanishing on star-commutators of two functions where one has compact support) for *all homogeneous* star products and hence all homogeneous star products are strongly closed in the sense of [12]. In particular the star products $*_{\mathrm{S}}$, $*_{\mathrm{W}}$, and the Fedosov star product $*_{\mathrm{F}}$ defined in [8] are strongly closed.

Lemma 8.1. *Let $D : C^\infty(T^*Q) \to C^\infty(T^*Q)$ be a homogeneous differential operator of degree $-r$ with $r \geq 1$, i.e. $[\mathcal{L}_\xi, D] = -rD$, then*

$$\int\limits_{T^*Q} D(g)\,\Omega = 0 \tag{66}$$

for all $g \in C_0^\infty(T^*Q)$ where $\Omega = (1/n!)\omega_0^{\wedge n}$ is the symplectic volume form.

Proof. First we consider a function g such that supp g is contained in the domain of a local bundle chart with coordinates $q^1, \ldots, q^n, p_1 \ldots, p_n$. Then due to the homogeneity D is locally given by

$$D(g) = \sum_{I,K \text{ with } |I| \geq r} D_I^K \frac{\partial^{|I|+|K|} g}{\partial p_I \partial q^K},$$

where we used the usual notation for multi-indices $I = (i_1, \ldots, i_{|I|})$. The homogeneity of D implies $\mathcal{L}_\xi D_I^K = (|I| - r) D_I^K$ for the coefficient functions D_I^K. Let $l := |I|$ and $I' := (i_1, \ldots, i_{l-1})$ then

$$\partial_{p_{i_l}} \left(D_I^K \frac{\partial^{|I|-1+|K|} g}{\partial p_{I'} \partial q^K} \right) = (\partial_{p_{i_l}} D_I^K) \frac{\partial^{|I|-1+|K|} g}{\partial p_{I'} \partial q^K} + D_I^K \frac{\partial^{|I|+|K|} g}{\partial p_I \partial q^K},$$

since $|I| \geq r \geq 1$. Hence we can conclude by induction on $l \geq r$ since for $l = r$ the first term vanishes due to the homogeneity of D_I^K that there exist smooth functions N_{iJ}^K defined in the domain of the chart such that

$$D_J^K \frac{\partial^{|I|+|K|} g}{\partial p_I \partial q^K} = \partial_{p_i} \left(\sum_{|J| \leq l-1} N_{iJ}^K \frac{\partial^{|J|+|K|} g}{\partial p_J \partial q^K} \right),$$

which clearly implies (66) for such g. Now let $g \in C_0^\infty(T^*Q)$ be arbitrary then using a partition of unity g can be decomposed into a finite sum of functions each having their support in the domain of a bundle chart. With the local argument from above the proof is completed. □

Corollary 8.2. *Let $*$ be a homogeneous star product for T^*Q and let $\widehat{T} \in C_{pp,k}^\infty(T^*Q)$ and $g \in C_0^\infty(T^*Q)$ then*

$$\int\limits_{T^*Q} \widehat{T} * g\,\Omega = \sum_{r=0}^{k} \lambda^r \int\limits_{T^*Q} C_r(\widehat{T}, g)\,\Omega \tag{67}$$

and analogously for $\int_{T^*Q} g * \widehat{T}\,\Omega$ where C_r denotes the bidifferential operator of $*$ in order λ^r.

Lemma 8.3. *Let $*$ be a homogeneous star product for T^*Q and $\widehat{T} \in C_{pp}^\infty(T^*Q)[[\lambda]]$ and $g \in C_0^\infty(T^*Q)[[\lambda]]$ then*

$$\int\limits_{T^*Q} \left(\widehat{T} * g - g * \widehat{T}\right) \Omega = 0. \tag{68}$$

Proof. Firstly we can assume that supp g is contained in the domain of a bundle chart and extend the statement afterwards by a partition of unity argument as in the proof of Lemma 8.1. Since in such a chart any $\widehat{T} \in C^\infty_{pp}(T^*Q)[\lambda]$ can be written locally as a $*$-polynomial in functions at most linear in the momenta with coefficient in $\mathbb{C}[\lambda]$ (due to [8, Prop. 3.7.ii]) it is sufficient to consider such polynomials. Clearly $\int_{T^*Q}(\pi^*\chi * g - g * \pi^*\chi)\, \Omega = 0$ for $\chi \in C^\infty(Q)$ and for $\widehat{X} \in C^\infty_{pp,1}(T^*Q)$ we obtain from (67)

$$\int\limits_{T^*Q} (\widehat{X} * g - g * \widehat{X})\, \Omega = \lambda \int\limits_{T^*Q} (C_1(\widehat{X}, g) - C_1(g, \widehat{X}))\, \Omega = \mathrm{i}\lambda \int\limits_{T^*Q} \{\widehat{X}, g\}\, \Omega = 0$$

since the integral over a Poisson bracket vanishes. Now a simple induction and the $\mathbb{C}[[\lambda]]$-linearity of $\int \ldots$ completes the proof. □

Finally we have to extend the statement of this lemma to arbitrary smooth functions:

Lemma 8.4. *Let D be a differential operator with compact support such that for any $\widehat{T} \in C^\infty_{pp}(T^*Q)$ the integral of $D(\widehat{T})\, \Omega$ over T^*Q vanishes. Then for any function $f \in C^\infty(T^*Q)$*

$$\int\limits_{T^*Q} D(f)\, \Omega = 0. \tag{69}$$

Proof. We shall prove this by induction on the order of D. If D is a differential operator of order 0 then it is just a left multiplication by a function D_0 having compact support. Now if $\int_{T^*Q} D_0 \widehat{T}\, \Omega = 0$ then $D_0 = 0$ since the functions polynomial in the momenta are uniformly dense on any compactum in all smooth functions due to the Stone–Weierstraß theorem. Now let D be a differential operator of order k and we may again firstly assume that supp D is contained in a local bundle chart and extend the result afterwards by a partition of unity. Writing $D(f) = \sum_{|I| \le k} D^I (\partial^{|I|} f / \partial x^I)$ we obtain by integration by parts that

$$\int\limits_{T^*Q} D(f)\, \Omega = \int\limits_{T^*Q} \tilde{D}(f)\, \Omega,$$

where

$$\tilde{D}(f) = -\sum_{|I|=k} \frac{\partial D^I}{\partial x_{i_k}} \frac{\partial^{|I|-1} f}{\partial x^{I'}} + \sum_{|I|<k} D^I \frac{\partial^{|I|} f}{\partial x^I}$$

is a differential operator of order $k-1$ and obviously $\int_{T^*Q} \tilde{D}(\widehat{T})\, \Omega = 0$ for all $\widehat{T} \in C^\infty_{pp}(T^*Q)$. Hence by induction the proof is complete. □

Collecting the results we obtain that any homogeneous star product on T^*Q is strongly closed:

Theorem 8.5. *Let $*$ be a homogeneous star product for T^*Q and $f, g \in C^\infty(T^*Q)[[\lambda]]$ where the coefficients of g have compact support. Then we have*

$$\int_{T^*Q} (f * g - g * f)\, \Omega = 0 \tag{70}$$

and hence any homogeneous star product is strongly closed, in particular $*_S$, $*_W$, and $*_F$.

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We would like to thank the old Babylonians for their trick to solve quadratic equations and Markus Pflaum for a fruitful discussion motivating us to add Section 8 of this paper.

Appendix A. Formal completed Newton–Puiseux series

In this appendix we shall briefly remember the definition and some basic facts about formal Laurent series, formal Newton–Puiseux series (NP series) and formal completed Newton–Puiseux series (CNP series). For proofs and further references we mention [9,28]. These formal series will generalize the formal power series in a natural way allowing more general exponents of the formal parameter. We define the allowed exponents in the following way: a subset $S \subset \mathbb{Q}$ is called L-admissable iff S has a smallest element and $S \subset \mathbb{Z}$, it is called NP-admissable iff it has a smallest element and there exists a natural number N such that $N \cdot S \subset \mathbb{Z}$, it is called CNP-admissable iff S has a smallest element and $S \cap [i, j]$ is finite for any $i, j \in \mathbb{Q}$. Now let K be a field, V a vector space over K and $f : \mathbb{Q} \to V$ a map, then we define the λ-support $\mathrm{supp}_\lambda f$ of f by $\mathrm{supp}_\lambda f := \{q \in \mathbb{Q} \mid f(q) \neq 0\}$. Then the formal Laurent series, the formal NP series and the formal CNP series all with coefficients in V are defined by

$$\begin{aligned} V((\lambda)) &:= \{f : \mathbb{Q} \to V \mid \mathrm{supp}_\lambda f \text{ is L-admissible } \}, \\ V\langle\langle\lambda^*\rangle\rangle &:= \{f : \mathbb{Q} \to V \mid \mathrm{supp}_\lambda f \text{ is NP-admissible } \}, \\ V\langle\langle\lambda\rangle\rangle &:= \{f : \mathbb{Q} \to V \mid \mathrm{supp}_\lambda f \text{ is CNP-admissible } \}. \end{aligned} \tag{A.1}$$

Note that clearly $V((\lambda)) \subseteq V\langle\langle\lambda^*\rangle\rangle \subseteq V\langle\langle\lambda\rangle\rangle$ are all K-vector spaces. Moreover, we denote by $V(\lambda)$ the subspace of $V((\lambda))$ of those elements with finite support and similarly $V\langle\lambda\rangle$. An element $f \in V\langle\langle\lambda\rangle\rangle$ is now usually written as formal series in the formal parameter λ

$$f = \sum_{q \in \mathrm{supp}_\lambda f} \lambda^q f_q, \quad \text{where } f_q := f(q). \tag{A.2}$$

Then the vector space structure of $V\langle\langle\lambda\rangle\rangle$ means just addition of the coefficients of the same power of λ and scalar multiplication of each coefficient and the vector space V itself can be identified with the subspace of those elements in $V\langle\langle\lambda\rangle\rangle$ with exponent 0 via the linear and injective map $V \ni v \mapsto \lambda^0 v \in V\langle\langle\lambda\rangle\rangle$. Note that $V\langle\langle\lambda\rangle\rangle$ and hence all its subspaces can be metrized by the following construction: we define for $0 \neq f \in V\langle\langle\lambda\rangle\rangle$ the order $\mathrm{o}(f) := \min(\mathrm{supp}_\lambda f)$ and set $\mathrm{o}(0) := +\infty$ and define $\varphi(f) := 2^{-\mathrm{o}(f)}$ resp. $\varphi(0) = 0$ and for $f, g \in V\langle\langle\lambda\rangle\rangle$ we set

$$d_\varphi(f, g) := \varphi(f - g), \tag{A.3}$$

which turns out to be an ultra-metric. Then $V[[\lambda]]$ and $V((\lambda))$ are known to be complete metric spaces with respect to d_φ and $V[\lambda]$ is dense in $V[[\lambda]]$ as well as $V(\lambda)$ is dense in $V((\lambda))$. Moreover, it was shown in [9] that $V\langle\langle\lambda\rangle\rangle$ is a complete metric space too and $V\langle\lambda\rangle$ as well as $V\langle\langle\lambda^*\rangle\rangle$ are dense subspaces. The topology induced by d_φ is usually called the λ-adic topology and the induced topology for $V \subseteq V\langle\langle\lambda\rangle\rangle$ is the discrete topology.

Next we consider an algebra $\mathcal{A}$ over K and define for $a, b \in \mathcal{A}\langle\langle\lambda\rangle\rangle$ a product by

$$ab = \left(\sum_{q \in \mathrm{supp}_\lambda a} \lambda^q a_q\right)\left(\sum_{p \in \mathrm{supp}_\lambda b} \lambda^p b_p\right) := \sum_{t \in \mathrm{supp}_\lambda a + \mathrm{supp}_\lambda b} \lambda^t \sum_{q+p=t} a_q b_p, \tag{A.4}$$

where $\mathrm{supp}_\lambda(ab) = \mathrm{supp}_\lambda a + \mathrm{supp}_\lambda b := \{q + p \mid q \in \mathrm{supp}_\lambda a, p \in \mathrm{supp}_\lambda b\}$ which turns out to be again a CNP-admissible subset of $\mathbb{Q}$. Note that in each order of λ the sum is finite and hence ab is again a well-defined element in $\mathcal{A}\langle\langle\lambda\rangle\rangle$. The following proposition is proved as in the case of formal power series:

Proposition A.1. *Let $\mathcal{A}$ be an algebra over K and let $\mathcal{A}\langle\langle\lambda\rangle\rangle$ be endowed with the product (A.4) then $\mathcal{A}\langle\langle\lambda\rangle\rangle$ is again a K-algebra which is associative resp. commutative resp. unital iff $\mathcal{A}$ is associative resp. commutative resp. unital. Moreover, $\mathcal{A}[[\lambda]] \subseteq \mathcal{A}((\lambda)) \subseteq \mathcal{A}\langle\langle\lambda^*\rangle\rangle$ are subalgebras of $\mathcal{A}\langle\langle\lambda\rangle\rangle$. Let in addition V be an $\mathcal{A}$-module then $V((\lambda))$ resp. $V\langle\langle\lambda^*\rangle\rangle$ resp. $V\langle\langle\lambda\rangle\rangle$ is an $\mathcal{A}((\lambda))$- resp. $\mathcal{A}\langle\langle\lambda^*\rangle\rangle$- resp. $\mathcal{A}\langle\langle\lambda\rangle\rangle$-module with the multiplication analogously to (A.4).*

In particular one can show that (A.4) applied for the field K itself defines on $\mathsf{K}((\lambda))$ resp. $\mathsf{K}\langle\langle\lambda^*\rangle\rangle$ resp. $\mathsf{K}\langle\langle\lambda\rangle\rangle$ again the structure of a field and in this case φ defines a non-archimedean and non-trivial absolute value for these fields. In the case when K is algebraically closed the Newton–Puiseux theorem ensures that $\mathsf{K}\langle\langle\lambda^*\rangle\rangle$ is again algebraically closed and by Kürschák's theorem one obtains that $\mathsf{K}\langle\langle\lambda\rangle\rangle$ is algebraically closed too and metric complete with respect to d_φ. These two features were the main motivation to consider formal CNP series instead of formal power series or formal Laurent series. As application of Proposition A.1 one obtains that $V((\lambda))$ resp. $V\langle\langle\lambda^*\rangle\rangle$ resp. $V\langle\langle\lambda\rangle\rangle$ are vector spaces over the fields $\mathsf{K}((\lambda))$ resp. $\mathsf{K}\langle\langle\lambda^*\rangle\rangle$ resp. $\mathsf{K}\langle\langle\lambda\rangle\rangle$.

In a next step we consider linear mappings between vector spaces of formal series. Let V and W be vector spaces over K and $\phi : V \to W$ a K-linear map. Then there exists a unique

$\mathsf{K}[[\lambda]]$-linear map $\tilde{\phi} : V[[\lambda]] \to W[[\lambda]]$ such that $\tilde{\phi}|V = \phi$ which is simply obtained by the '$\mathsf{K}[[\lambda]]$-linear continuation' of ϕ. The same result is true for formal Laurent, NP, and CNP series and for simplicity we shall always identify ϕ with its continuation and use the same symbol. Using this identification we get the following natural inclusions:

$$\begin{aligned}
&\mathsf{Hom}_{\mathsf{K}}(V, W)[[\lambda]] \subseteq \mathsf{Hom}_{\mathsf{K}[[\lambda]]}(V[[\lambda]], W[[\lambda]]),\\
&\mathsf{Hom}_{\mathsf{K}}(V, W)((\lambda)) \subseteq \mathsf{Hom}_{\mathsf{K}((\lambda))}(V((\lambda)), W((\lambda))),\\
&\mathsf{Hom}_{\mathsf{K}}(V, W)\langle\langle\lambda^*\rangle\rangle \subseteq \mathsf{Hom}_{\mathsf{K}\langle\langle\lambda^*\rangle\rangle}(V\langle\langle\lambda^*\rangle\rangle, W\langle\langle\lambda^*\rangle\rangle),\\
&\mathsf{Hom}_{\mathsf{K}}(V, W)\langle\langle\lambda\rangle\rangle \subseteq \mathsf{Hom}_{\mathsf{K}\langle\langle\lambda\rangle\rangle}(V\langle\langle\lambda\rangle\rangle, W\langle\langle\lambda\rangle\rangle).
\end{aligned} \tag{A.5}$$

Similar inclusions are valid for multilinear maps as well. In the case of formal power series it is known that any $\mathsf{K}[[\lambda]]$-linear map $\phi : V[[\lambda]] \to W[[\lambda]]$ is of the form $\phi = \sum_{r=0}^{\infty} \lambda^r \phi_r$ with $\phi_r \in \mathsf{Hom}_{\mathsf{K}}(V, W)$ and hence the first inclusion in (A.5) is in fact an equality [15, Prop. 2.1]. This is in general no longer the case for the other three inclusions: let $\dim V = \infty$ and $\dim W > 0$ and let $\{e_n\}_{n\in\mathbb{N}}$ be a set of linear independent vectors in V and hence they are still linear independent over $\mathsf{K}((\lambda))$ in $V((\lambda))$. Complete this set to a base of $V((\lambda))$ by some vectors $\{f_\nu\}_{\nu\in I}$ and define a $\mathsf{K}((\lambda))$-linear map $\phi : V((\lambda)) \to W((\lambda))$ by $\phi(e_n) := \lambda^{-n} w$ and $\phi(f_\nu) := 0$ where $0 \neq w \in W$ is some chosen vector. Then clearly $\phi \notin \mathsf{Hom}_{\mathsf{K}}(V, W)((\lambda))$. Nevertheless we are mainly interested in those linear maps which can be written as formal series in K-linear maps.

Now we consider certain continuous functions between vector spaces of formal series: let V, W be K-vector spaces and let $T : V\langle\langle\lambda\rangle\rangle \to W\langle\langle\lambda\rangle\rangle$ be a not necessarily linear map such that there exists a $q \in \mathbb{Q}$ such that

$$\mathrm{o}(T(v) - T(v')) \geq \mathrm{o}(v - v') + q \tag{A.6}$$

for all $v, v' \in V\langle\langle\lambda\rangle\rangle$. Then we say that T raises the degree at least by q. For such maps we have the following property which is proved straightforward:

Lemma A.2. *Let $T : V\langle\langle\lambda\rangle\rangle \to W\langle\langle\lambda\rangle\rangle$ be a map raising the degree at least by $q \in \mathbb{Q}$. Then T is Lipschitz-continuous with respect to the metric d_φ with Lipschitz-constant 2^{-q}. In particular those $\mathsf{K}\langle\langle\lambda\rangle\rangle$-linear maps of the form $\phi = \sum_{q\in\mathrm{supp}_\lambda \phi} \lambda^q \phi_q \in \mathsf{Hom}_{\mathsf{K}}(V, W)\langle\langle\lambda\rangle\rangle$ are Lipschitz-continuous with Lipschitz-constant $2^{-\min(\mathrm{supp}_\lambda \phi)}$. The same is true for formal power, Laurent, and NP series.*

Then a nice consequence of the metric completeness of $V[[\lambda]]$, $V((\lambda))$, and $V\langle\langle\lambda\rangle\rangle$ is the following formal version of Banach's fixed point theorem:

Proposition A.3 (Formal Banach's fixed point theorem). *Let K be a field and V a vector space over K and $T : V\langle\langle\lambda\rangle\rangle \to V\langle\langle\lambda\rangle\rangle$ a map raising the degree at least by $q > 0$. Then there exists a unique fixed point $v_0 \in V$ of T, i.e.*

$$T(v_0) = v_0 \tag{A.7}$$

and v_0 can be obtained by $v_0 = \lim_{n\to\infty} T^n(v)$ where $v \in V\langle\langle\lambda\rangle\rangle$ is arbitrary and the limit is with respect to the λ-adic topology. The same result holds for formal power series and Laurent series.

Note that the proposition is not true in general for the formal NP series since $V\langle\langle\lambda^*\rangle\rangle$ is not Cauchy-complete. The above proposition is useful in many situations in deformation quantization where one has to 'solve an equation by recursion', e.g. in the construction of Fedosov star products or for the construction of the time development operator (see e.g. Appendix B).

In the following we shall consider the possibility to extend several structures which are defined for formal power series to formal Laurent, NP, and CNP series. In particular, we are interested in algebra deformations in the sense of Gerstenhaber (see e.g. [20]): Let $(\mathcal{A}, \mu_0)$ be an algebra over K and let $\mu_i : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be bilinear maps for $i \geq 1$. Then we consider the formal deformation $(\mathcal{A}[[\lambda]], \mu)$ where $\mu := \sum_{r=0}^{\infty} \lambda^r \mu_r$ as a module over $\mathsf{K}[[\lambda]]$. (Usually one has additional conditions for the deformation μ, for example it should be associative if μ_0 is associtive etc.)

Lemma A.4. *Let $(\mathcal{A}, \mu_0)$ be an algebra over K and let $(\mathcal{A}[[\lambda]], \mu = \sum_{r=0}^{\infty} \lambda^r \mu_r)$ be a deformation of $(\mathcal{A}, \mu_0)$. Then $(\mathcal{A}((\lambda)), \mu)$ resp. $(\mathcal{A}\langle\langle\lambda^*\rangle\rangle, \mu)$ resp. $(\mathcal{A}\langle\langle\lambda\rangle\rangle, \mu)$ are algebras over $\mathsf{K}((\lambda))$ resp. $\mathsf{K}\langle\langle\lambda^*\rangle\rangle$ resp. $\mathsf{K}\langle\langle\lambda\rangle\rangle$ which are associative resp. Lie resp. commutative iff $(\mathcal{A}[[\lambda]], \mu)$ is associative resp. Lie resp. commutative. Let $(\mathcal{B}[[\lambda]], \tilde{\mu})$ be another deformed algebra and $\phi : \mathcal{A}[[\lambda]] \to \mathcal{B}[[\lambda]]$ a $\mathsf{K}[[\lambda]]$-linear map. Then ϕ is an (anti-) homomorphism of $\mathsf{K}[[\lambda]]$-modules iff the extension $\phi : (\mathcal{A}((\lambda)), \mu) \to (\mathcal{B}((\lambda)), \tilde{\mu})$ resp. $\phi : (\mathcal{A}\langle\langle\lambda^*\rangle\rangle, \mu) \to (\mathcal{B}\langle\langle\lambda^*\rangle\rangle, \tilde{\mu})$ resp. $\phi : (\mathcal{A}\langle\langle\lambda\rangle\rangle, \mu) \to (\mathcal{B}\langle\langle\lambda\rangle\rangle, \tilde{\mu})$ is an (anti-) homomorphism of $\mathsf{K}((\lambda))$- resp. $\mathsf{K}\langle\langle\lambda^*\rangle\rangle$- resp. $\mathsf{K}\langle\langle\lambda\rangle\rangle$-algebras. The same is true for a $\mathsf{K}[[\lambda]]$-linear derivation D of $(\mathcal{A}[[\lambda]], \mu)$.*

This easy lemma is very useful since it justifies to perform many calculation only in the setting of formal power series and extend the results afterwards.

A last important concept in particular for the general GNS construction as proposed in [9] is the notion of positivity. Here we consider an ordered field R and a quadratic field extension $\mathsf{C} := \mathsf{R}(\mathbf{i})$ where $\mathbf{i}^2 := -1$ (of course we have in mind to use the fields of real and complex numbers). Then $\mathsf{R}[[\lambda]]$ is an ordered ring and $\mathsf{R}((\lambda))$, $\mathsf{R}\langle\langle\lambda^*\rangle\rangle$, and $\mathsf{R}\langle\langle\lambda\rangle\rangle$ are ordered fields by the following definition: an element $a = \sum_{q\in \mathrm{supp}_\lambda a} \lambda^q a_q$ is called positive iff $a_{q_0} > 0$ in R where $q_0 := \min(\mathrm{supp}_\lambda a)$. Complex conjugation in C is defined as usual and extended to $\mathsf{C}[[\lambda]]$, $\mathsf{C}((\lambda))$, $\mathsf{C}\langle\langle\lambda^*\rangle\rangle$, and $\mathsf{C}\langle\langle\lambda\rangle\rangle$ by the definition that the formal parameter (and all its powers) should be real: $\overline{\lambda} := \lambda$. Note that the topology induced by the order coincides with the metric topology induced by d_φ.

Now let $(\mathcal{A}, \mu_0)$ be an associative C-algebra and $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r$ an associative deformation of μ_0. Let moreover $^* : \mathcal{A}[[\lambda]] \to \mathcal{A}[[\lambda]]$ be an involutive $\mathsf{C}[[\lambda]]$-antilinear anti-automorphism of the form $^* = \sum_{r=0}^{\infty} \lambda^r {}^{*_r} \in \mathrm{Hom}_{\mathsf{K}}(\mathcal{A}, \mathcal{A})[[\lambda]]$. Then a $\mathsf{C}[[\lambda]]$-linear functional $\omega : \mathcal{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ is called positive iff for all $A \in \mathcal{A}[[\lambda]]$ we have $\omega(\mu(A^*, A)) \geq 0$ in $\mathsf{R}[[\lambda]] \subset \mathsf{C}[[\lambda]]$ and analogously for formal Laurent, NP, and CNP series. Then the following lemma shows that the positivity of a linear functional is already determined by the elements in $\mathcal{A}$:

Lemma A.5. *Let $\mathcal{A}$, μ and * be as above and let $\omega : \mathcal{A}\langle\langle\lambda\rangle\rangle \to \mathsf{C}\langle\langle\lambda\rangle\rangle$ be a $\mathsf{C}\langle\langle\lambda\rangle\rangle$-linear functional of the form $\omega = \sum_{q\in\mathrm{supp}_\lambda\,\omega} \lambda^q \omega_q \in (\mathcal{A}^*)\langle\langle\lambda\rangle\rangle$ (where $\mathcal{A}^*$ denotes the algebraic dual of $\mathcal{A}$) such that for all $A \in \mathcal{A}$ one has $\omega(\mu(A^*, A)) \geq 0$. Then ω is a positive functional on $\mathcal{A}\langle\langle\lambda\rangle\rangle$. The analogous result holds for formal power, Laurent, and NP series.*

Proof. Let $A = \sum_{q\in\mathrm{supp}_\lambda\, A} \lambda^q A_q \in \mathcal{A}\langle\langle\lambda\rangle\rangle$ then we have to prove $\omega(A^*A) \geq 0$. Using the Cauchy–Schwartz inequality for ω applied for $A_q \in \mathcal{A}$ which is valid due to the assumption we notice that if $\omega(A_q^* A_q) = 0$ then none of the terms involving A_q in $\omega(A^*A)$ contributes. Hence we can assume that $\omega(A_q^*A_q) > 0$ for all $q \in \mathrm{supp}_\lambda A$. Now let $\mathrm{supp}_\lambda A = \{q_0 < q_1 < \cdots\}$ and we can furthermore assume $q_0 = 0$. Let $c := \omega(A_0^*A_0) > 0$, $a := \omega(A_{q_1}^*A_{q_1}) > 0$ and $b := \omega(A_0^*A_{q_1})$ then the positivity of ω applied for A_0 and A_{q_1} implies $\omega(A_{q_1}^*A_0) = \overline{b}$ and $b\overline{b} \leq ac$. This implies on the other hand that $\omega((A_0 + tA_{q_1})^*(A_0 + tA_{q_1})) = at^2 + (b + \overline{b})t + c$ is non-negative for all $t = \overline{t} \in \mathsf{R}\langle\langle\lambda\rangle\rangle$ and hence it is non-negative for $t = \lambda^{q_1}$. Now one proceeds analogously by induction to obtain that ω is positive for all finite sums $A_N := \sum_{q\leq N} \lambda^q A_q$ but then the continuity of μ, * and ω according to Lemma A.2 guarantees that ω is in fact a positive linear functional on all $A \in \mathcal{A}\langle\langle\lambda\rangle\rangle$ since the order topology for $\mathsf{R}\langle\langle\lambda\rangle\rangle$ coincides with the metric topology induced by d_φ. □

Appendix B. Time development in deformation quantization

In this appendix we shall briefly remember some well-known facts about the time development in deformation quantization (see e.g. [9, Sec. 5] and Fedosov's book [19, Sec. 5.4]). Let (M, ω) be a symplectic manifold and let $*$ be a star product for $C^\infty(M)[[\lambda]]$. Let X be a symplectic vector field with complete flow ϕ_t and $\beta = i_X\omega$ the corresponding closed but not necessarily exact one-form. Then locally β is exact and we have $\beta = \mathrm{d}H$. We use this locally defined Hamiltonian H to define star product commutators locally and notice that $\mathrm{ad}(H)$ does not depend on the choice of H but only on β and hence one obtains a globally defined map which we shall denote by $\mathrm{ad}(\beta)$. Then we consider the Heisenberg equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\mathrm{i}}{\lambda}\mathrm{ad}(\beta) f(t) \tag{B.1}$$

with respect to β and ask for a solution $t \mapsto f(t) \in C^\infty(M)[[\lambda]]$ for a given initial value $f(0)$:

Theorem B.1. *Let X be a symplectic vector field with complete flow and let $\beta = i_X\omega$. Then the Heisenberg equation of motion (B.1) has a unique solution $f(t)$ for any initial value $f(0) \in C^\infty(M)[[\lambda]]$ and $t \in \mathbb{R}$.*

Proof. Eq. (B.1) is equivalent to the equation $\mathrm{d}/\mathrm{d}t\, g(t) = \phi^*_{-t} \circ \hat{H} \circ \phi^*_t g(t)$ where $\hat{H} := (\mathbf{i}/\lambda)\mathrm{ad}(\beta) - \mathcal{L}_X$ and $g(t) = \phi^*_{-t} f(t)$. This equation can be rewritten as a fixed point equation by integrating over t. Then Proposition A.3 shows that there exists a unique solution since $\hat{H}$ raises the λ-degree at least by one. □

This theorem allows us to define a quantum mechanical time development operator A_t for the symplectic vector field X such that $f(t) = A_t f$ is the unique solution with initial condition f. Then A_t is a $\mathbb{C}[[\lambda]]$-linear map for all $t \in \mathbb{R}$ and clearly $A_0 = \mathsf{id}$. Moreover, A_t clearly satisfies the Heisenberg equation as operator equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A_t = \frac{\mathbf{i}}{\lambda} \mathrm{ad}(\beta) A_t. \tag{B.2}$$

The quantum mechanical time development operator A_t is obtained by a quantum correction of the classical time development operator which is just ϕ^*_t. We denote this correction by

$$T_t := \phi^*_{-t} \circ A_t \tag{B.3}$$

and prove the following proposition by a straightforward computation:

Proposition B.2. *The operator T_t is a formal power series of differential operators $T_t = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r T_t^{(r)}$ and T_t satisfies the following differential equation with initial condition $T_0 = \mathsf{id}$*

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = \phi^*_{-t} \circ \hat{H} \circ \phi^*_t \circ T_t \tag{B.4}$$

and the equivalent integral equation

$$T_t = \mathsf{id} + \int_0^t \phi^*_{-\tau} \circ \hat{H} \circ \phi^*_\tau \circ T_\tau \, \mathrm{d}\tau \tag{B.5}$$

where $\hat{H} = (\mathbf{i}/\lambda)\mathrm{ad}(\beta) - \mathcal{L}_X$ is defined as in the proof of Theorem B.1. *If the star product is of the Vey type then $T_t^{(r)}$ is a differential operator of order $2r$.*

In a last step we prove that A_t is a one-parameter group of automorphisms of the star product using the fact that the solution $f(t)$ of (B.1) is uniquely determined by $f(0)$:

Theorem B.3. *The quantum mechanical time development operator A_t of the symplectic vector field X with complete classical flow has the following properties where $\beta = i_X\omega$:*

(i) $A_t A_s = A_{t+s} = A_s A_t$ *and* $A_0 = \mathsf{id}$ *for all* $t, s \in \mathbb{R}$.
(ii) $A_t \mathrm{ad}(\beta) = \mathrm{ad}(\beta) A_t$ *for all* $t \in \mathbb{R}$.
(iii) $A_t(f * g) = A_t f * A_t g$ *for all* $f, g \in C^\infty(M)[[\lambda]]$ *and* $t \in \mathbb{R}$.
(iv) A_{-t} *is the time development operator for the vector field* $-X$ *and* $(T_t)^{-1} = \phi^*_{-t} \circ T_{-t} \circ \phi^*_t$.
(v) *If in addition* $\overline{f * g} = \overline{g} * \overline{f}$ *then* A_t *is a real automorphism, i.e.* $\overline{A_t f} = A_t \overline{f}$.

Remark. All statements are still correct if one replaces the closed one-form β by $\beta + \sum_{r=1}^{\infty} \lambda^r \beta_r$ where β_r are again closed one-forms which enclose the case of some 'quantum corrections'. Furthermore all statements can be transferred to the case of formal Laurent and CNP series including the corresponding quantum corrections (but not necessarily to formal NP series since we used the formal Banach's fixed point theorem).

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